Change of basis in Numeration Systems

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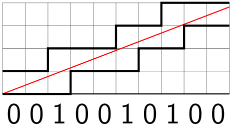
Based on joint work with

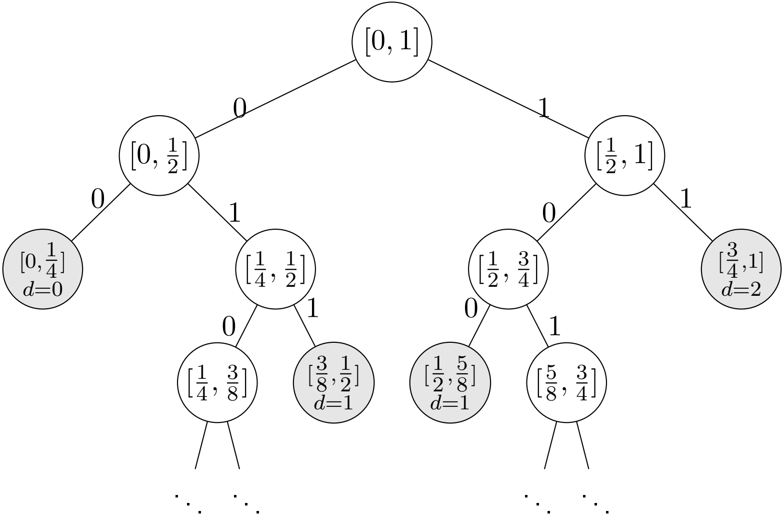
Valérie Berthé, Eda Cesaratto and Martín D. Safe



Meeting EPA!,

Buenos Aires, 23 October, 2024.

Example: first digit d in base 3 of $x \in [0,1)$ from binary $x = (0.b_1 b_2 \ldots)_2$?



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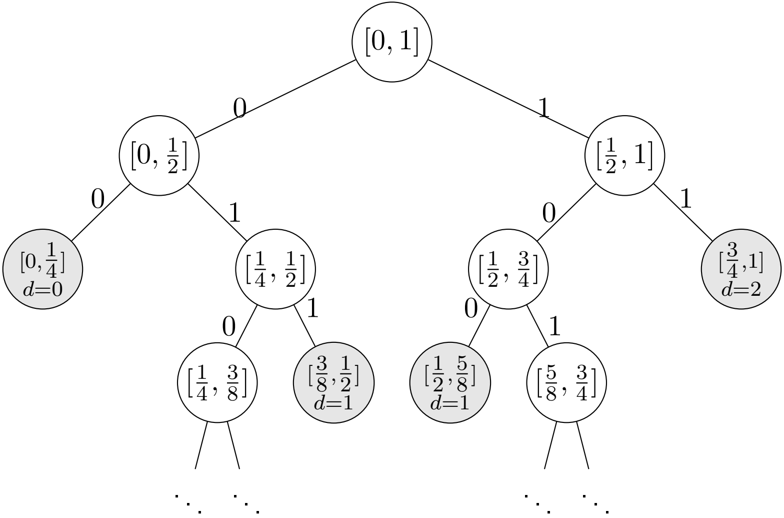

$\Rightarrow$ Only intervals containing $\frac{1}{3}$ and $\frac{2}{3}$ remain. Expected cost

$$\mathbb{E}[C] = \sum_{k \geq 0} \Pr(C > k) = 1 + \sum_{k \geq 1} \frac{2}{2^k} = 3 \texttt{ bits}.$$

Change of basis: binary to d-ary

- Given n binary digits $b_1, b_2, \ldots, b_n \in \{0,1\}$ of
$$x = (0.b_1 b_2 \ldots)_2 \in [0,1] .$$
- Number $L = L_n(x)$ of d-ary digits $0 \leq d_1, \ldots, d_L < d$ deduced?
$$x = (0.d_1 d_2 \ldots)_d \in [0,1] .$$

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- For $d = 2^A$ we simply obtain

$$L_n(x) = n/A \, ,$$

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  One digit in base d^L "corresponds" to one in base 2^n if $d^L \approx 2^n$.

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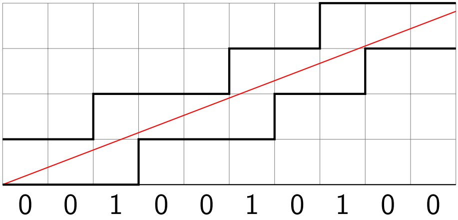
Motivation: simulating Sturmian words

Sturmian words. discrete coding of lines: horizontal (0), vertical (1) steps



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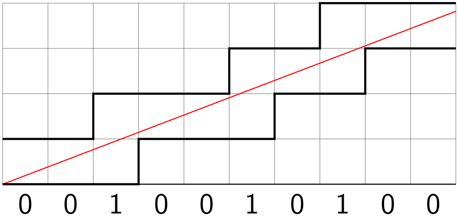

Theorem (Morse, Hedlund '40)

Binary sequence (u_k) *is Sturmian iff there is an irrational* $\alpha \in (0,1)$ *and* $\beta \in [0,1)$ *such that for all* $k \geq 0$*,*

$$u_k = \lfloor (k+1)\alpha + \beta \rfloor - \lfloor k\alpha + \beta \rfloor .$$

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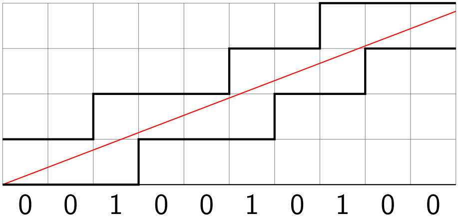

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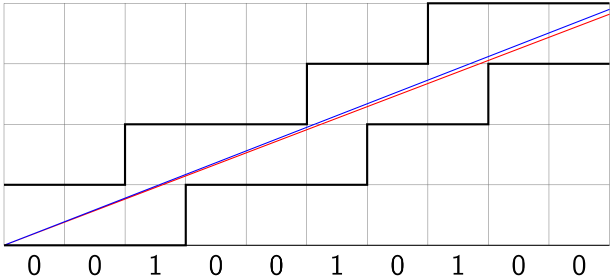
Remark The parameters $\alpha \in [0,1) \setminus \mathbb{Q}$ and $\beta \in [0,1)$ are unique !

Question. if we have approximation of α, and $\beta = 0^{\dagger}$, how many *Sturm* digits (u_k) of α are deduced?



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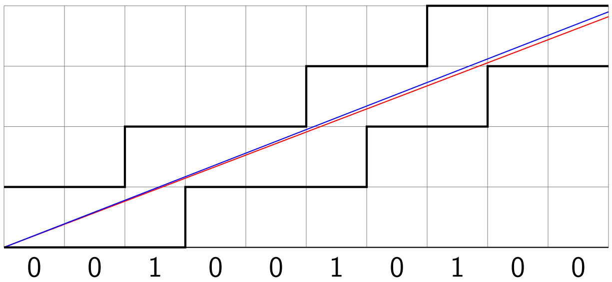

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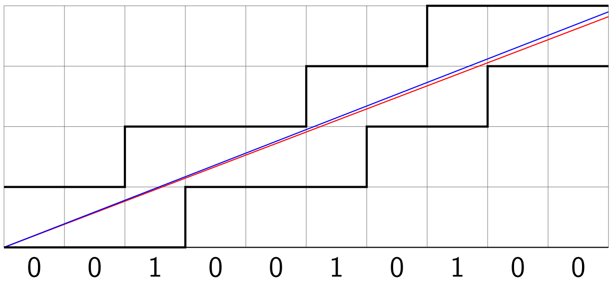

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$$\Longrightarrow \text{rational } a/b \in [\alpha_2, \alpha_1] \text{ implies } u_{b-1}^{\langle \alpha_2 \rangle} = 0, u_{b-1}^{\langle \alpha_1 \rangle} = 1.$$

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Plan of the talk

1. Unidimensional partitions of positive entropy
2. Undimensional partitions with zero entropy
3. Farey partition: zero entropy partitions for Sturmian digits
4. Bidimensional partitions
5. Conclusions and other work

Section

First historical results: Lochs' Theorem

- Given n decimal digits $d_1, d_2, \ldots, d_n$ of $x \in [0,1]$,
$$x = (0.d_1 d_2 \ldots)_{10} \in [0,1]\,.$$
- Number $L_n(x)$ of CFE-digits (partial quotients) deduced without error ?
$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}\,.$$

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Theorem (Lochs '64)

The rate of CF-digits per decimal given satisfies

$$\lim_{\mathtt{d}\to\infty} \frac{L_n(x)}{n} = \frac{6\log 2 \log 10}{\pi^2} \doteq 0.9702701\ldots\,,$$

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"Example". The first 1000 decimals of π determine exactly 968 partial quotients of π.

Systems of partitions: a model for numeration

Example: decimal expansion

Associated partitions $\mathcal{D} = (\mathcal{D}_n)$ for the decimal expansion:

$$\mathcal{D}_n = \left\{ \left(\frac{k}{10^n}, \frac{k+1}{10^n} \right) : k \in \{0, 1, \ldots, 10^n - 1\} \right\}.$$

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Intervals determine expansion up to depth n:

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Definition (System of interval partitions)

Sequence of (open) interval partitions $\mathcal{P} = (\mathcal{P}_n)$ of $[0, 1]$

- $\mathcal{P}_{n+1}$ refinement of $\mathcal{P}_n$ for every n.
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Model for numeration systems: more generally,

- notation $I_n^{\mathcal{P}}(x) = I \in \mathcal{P}_n$ such that $x \in I$,
- first n symbols for x determine $I_n^{\mathcal{P}}(x)$ and conversely.

Entropy of a partition

Entropy dictates size of intervals

- *Shannon entropy*[‡]:

$$H(\mathcal{P}) = -\lim_{k\to\infty}\frac{1}{k}\sum_{I\in\mathcal{P}_k}|I|\log|I|\ .$$

- *Point-wise entropy*: for almost every x

$$h(\mathcal{P}) = -\lim_{k\to\infty}\frac{1}{k}\log\left|I_k^{\mathcal{P}}(x)\right|\ .$$

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Remark. By Fatou's Lemma $h(\mathcal{P}) \leq H(\mathcal{P})$ if both exist.

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Existence of point-wise entropy

Systems of partitions associated with good (positive entropy) dynamical systems have point-wise entropy:

Theorem (Shannon,McMillan,Breiman)

Let T be an ergodic measure preserving transformation on a probability space $(\Omega, \mathcal{B}, \mu)$ and let P be a finite or countable generating partition for T for which $H_\mu(P) < \infty$. Then for μ-a.e. x,

$$\lim_{n\to\infty} -\frac{\log \mu\left(P_n(x)\right)}{n} = h_\mu(T)\,.$$

Here $H_\mu(P)$ denotes the entropy of the partition P, $h_\mu(T)$ the entropy of T and $P_n(x)$ denotes the element of the partition $\bigvee_{i=0}^{n-1} T^{-i}P$ containing x.

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Lochs' index for systems of partitions $\mathcal{P}^1, \mathcal{P}^2$

$$L_n(x; \mathcal{P}^1, \mathcal{P}^2) := \sup\{m \geq 0 : I_n^{\mathcal{P}^1}(x) \subset I_m^{\mathcal{P}^2}(x)\},$$

depth in $\mathcal{P}^2$ deduced from depth n in $\mathcal{P}^1$.

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Explanation

If $I_n^{\mathcal{P}^1}(x)$ splits over (intersects) several $J \in \mathcal{P}_m^2$,

$\Longrightarrow$ we cannot yet decide on $I_m^{\mathcal{P}^2}(x)$

Unidimensional partitions of positive entropy

Theorem (Dajani, Fieldsteel, 2001)

Consider systems of partitions $\mathcal{P}^1$ and $\mathcal{P}^2$, with positive point-wise entropies $h(\mathcal{P}^1)$ and $h(\mathcal{P}^2)$. Then

$$\lim_{n\to\infty} \frac{1}{n} L_n(x; \mathcal{P}^1, \mathcal{P}^2) = \frac{h(\mathcal{P}^1)}{h(\mathcal{P}^2)}$$

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We deduce Lochs' Theorem and result for d-ary basis:

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- **Base d.** Since $|I_n^{\mathcal{D}}(x)| = d^{-n}$, $h(\mathcal{D}) = \log d$.
- **Continued fractions.** Entropy $h(\mathcal{C}) = \frac{\pi^2}{6 \log 2}$

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What if $h(\mathcal{P}_1) = 0$ or $h(\mathcal{P}_2) = 0$? e.g., Sturm digits (u_k)

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- If $h(\mathcal{P}_2) > 0$ and $h(\mathcal{P}_1) = 0$, almost surely $L/t \to 0$.

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In our work [BCRS'23] we generalize this result to zero entropy...

Section

Log-balancedness and weight function

Definition (Weight function)

A system of partitions $\mathcal{P} = (\mathcal{P}_n)$ is *log-balanced* a.e. (resp. in measure) with *weight function* $f\colon \mathbb{N} \to \mathbb{R}_{>0}$, $f(n) \to \infty$, if

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Example

- For positive entropy $h = h(\mathcal{P}) > 0$

$$f(n) = h \times n\,.$$

- If partition is log-balanced, entropy 0 corresponds to

$$f(n) = o(n)\,.$$

Result for zero entropy

Theorem (Berthé,Cesaratto,R.,Safe, 2023)

Consider systems of partitions $\mathcal{P}^1$ *and* $\mathcal{P}^2$*, with a.e. weight functions* f_1 *and* f_2*. Then, under certain technical conditions*

$$\lim_{n\to\infty} \frac{f_2\left(L_n(x;\mathcal{P}^1,\mathcal{P}^2)\right)}{f_1(n)} = 1\,,$$

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The conditions are:

- $\sum e^{-\delta f_1(n)} < \infty$ for every $\delta > 0$;
- f_2 is non decreasing ;
- $f_2(n+1) - f_2(n) = o(f_2(n))$ as $n \to \infty$.

Discussion: conditions of our main result

We recall the conditions:

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Intuitively, the first condition is the most constraining one:

- Condition (b) reflects the fact that $\mathcal{P}_2$ is refining ;
- Condition (c) means that $f_2(n+1) \sim f_2(n)$;
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Discussion: conditions of our result for zero entropy

Example: appropriate output partitions $\mathcal{P}_2$

Subexponential weight functions of the form

$$f_2(n) = \exp(g(n)),$$

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$$f_1(n) = (\log n) \cdot g(n)\,,$$

with $g(t) \to \infty$.

Section

1. Unidimensional partitions of positive entropy

2. Undimensional partitions with zero entropy

3. Farey partition: zero entropy partitions for Sturmian digits

4. Bidimensional partitions

5. Conclusions and other work

A zero entropy system for Sturmian digits

Farey partition (Sturm source) is built by splitting intervals at *mediant*

$$\texttt{mediant}(a/b, c/d) := (a+b)/(c+d)\,.$$

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Construction of the *Farey partition* $\mathcal{F}_n$:

- Base case: $\mathcal{F}_0 = \{[0,1]\}$.
- Building $\mathcal{F}_n$: split $\left[\frac{a}{b}, \frac{c}{d}\right] \in \mathcal{F}_{n-1}$ at mediant $\frac{a+c}{b+d}$, if $b+d \leq n+1$.

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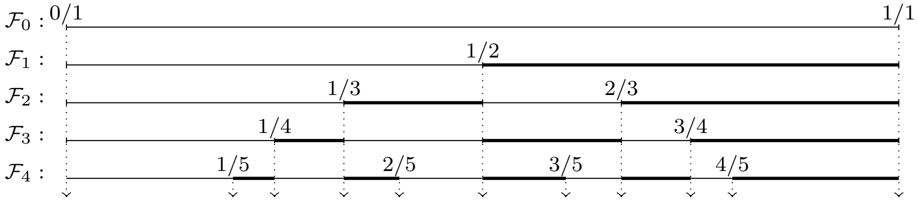

Farey partition: natural zero entropy system with weight

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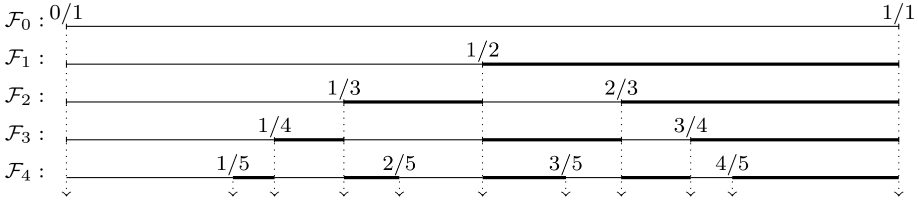

Properties:

- $\mathcal{F}_k$ determines[§] char. Sturmian word up to u_k: prefix $u_0 \ldots u_k$.
- The end-points $\mathcal{F}_k$ are exactly $\{\frac{a}{b} \in \mathbb{Q} : 0 \leq a \leq b \leq k+1\}$.
- Small number: $\Theta(k^2)$ intervals in $\mathcal{F}_k$

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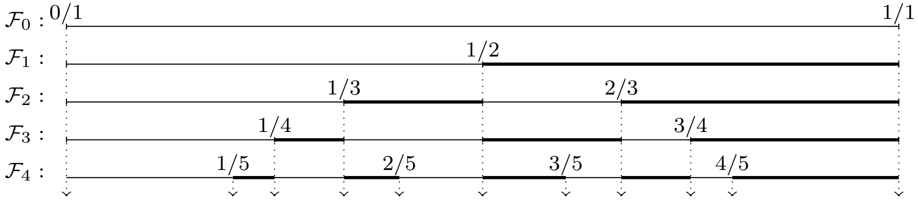

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Weight of the Farey partition

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Farey intervals have comparable size almost everywhere:

Lemma

For almost every x, for large $n \geq n_0(x)$

$$\frac{1}{n^2} \leq \left|I_n^{\mathcal{F}}(x)\right| \leq \frac{(\log n)(\log\log n)}{n^2}$$

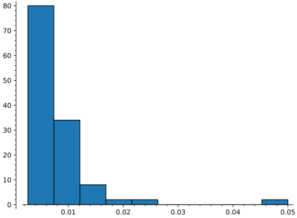

Figure. Histogram of interval sizes for $n = 20$. $\frac{1}{20^2} = 0.0025$, $\frac{1}{20} = 0.05$.

Consequences: producing digits of Sturmian word

From n digits of the slope α, we deduce *exponentially many*:

Corollary: from binary to Farey

Let $\mathcal{F}$ be the Farey partition, then

$$\log L_n(x; \mathcal{B}, \mathcal{F}) \sim \frac{\log 2}{2} \times n\,,$$

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Corollary

Let $\mathcal{P}$ with $h(\mathcal{P}) > 0$ and $\mathcal{F}$ be the Farey partition, then

$$\log L_n(x; \mathcal{P}, \mathcal{F}) \sim \frac{h(\mathcal{P})}{2} \times n\,,$$

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Section

Two dimensions: joint coding of a pair

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Theorem (Dajani, De Vries, Johnson 2005)

Consider systems of partitions $\mathcal{P}^1$ and $\mathcal{P}^2$ of the square $[0,1]^2$ satisfying

(i) *$\mathcal{P}^1$ is made out of squares.*

(ii) *$\mathcal{P}^2$ consisting of convex polygons, of pointwise entropy $h(\mathcal{P}^2) > 0$.*

(iii) *There are constants $\beta, c_0, c_1 > 0$ so that, for every I from a partition in $\mathcal{P}^2$, $c_0\lambda(I) \leq (\text{diam}(I))^\beta \leq c_1\lambda(I)$.*

Then, for a.e. $(x,y) \in [0,1]^2$,

$$\lim_{n\to\infty} \frac{1}{n} L_n(x,y;\mathcal{P}^1,\mathcal{P}^2) = \frac{\beta}{2(\beta-1)} \frac{h(\mathcal{P}^1)}{h(\mathcal{P}^2)}\,.$$

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Example of interest: Ostrowski expansion

Ostrowski transformation

Given irrationals $x, y \in [0, 1]$ define

$$S(x, y) = (\{1/x\}, \{y/x\}) ,$$

where $\{t\} := t - \lfloor t \rfloor$ is the fractional part.

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Digits are produced at each iteration $i \geq 1$ by (x_i, y_i)

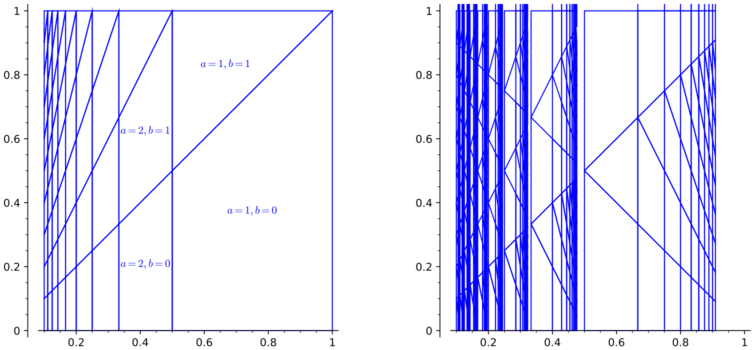
$$a_i = \lfloor 1/x_i \rfloor , \qquad b_i = \lfloor y_i/x_i \rfloor .$$

We retrieve (x, y) by

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}, \qquad y = \sum_{i=1}^{\infty} b_i \cdot x_0 \ldots x_{i-1} .$$

Partitions: Ostrowski expansion

Partition $\mathcal{P}_1$ according to (a_1, b_1) and $\mathcal{P}_2$ according to (a_1, b_1, a_2, b_2).



Section

Final remarks and conclusions

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   ⇒ measure and diameter not enough.

References

Thank you for your attention! 

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