Absorbing patterns in BST-like expression-trees

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Joint work with

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LIPN, Sorbonne Paris Nord

Meeting EPA!,
Buenos Aires, 24 October, 2024.

Plan of the talk

1. Random BST-like tree model
2. Semantic simplifications
3. Result for BST-like trees and elements of the proof
4. Conclusions

Section

1. Random BST-like tree model

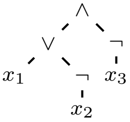
2. Semantic simplifications

3. Result for BST-like trees and elements of the proof

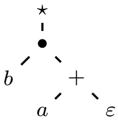
4. Conclusions

Introduction

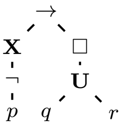
- Expression trees



$(x_1 \vee \neg x_2) \wedge \neg x_3$



$(b \cdot (a + \varepsilon))^\star$



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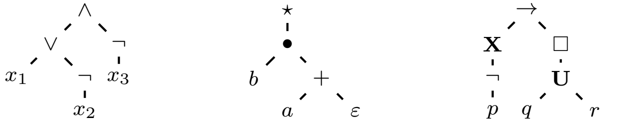

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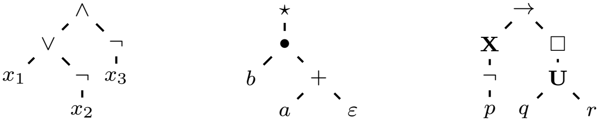

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  - Correctness and performance of algorithms

- Randomly generated input
  - Realistic distribution
  - Simple implementation, possibility of theoretical analysis.

BST-like trees: a natural construction algorithm

Problem: sample random tree of size n,

- representing expression with unary and binary operators,
- leaves correspond to constants or variables.

[1] If $n = 2$, constrained to arity one.

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Example.

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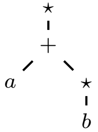

$\Bigg) = p_\star p_+ \frac{1}{2} p_a p_b$

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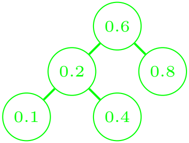
Why binary search "like" ?

- Build BST from n random numbers $u_i \in [0, 1]$:



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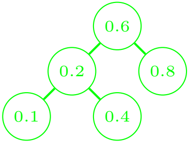

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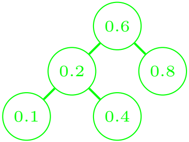

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Same construction: force our subtrees to have $|T_L|, |T_R| \geq 1$, as node corresponds to binary operator.

BST-like trees: a natural construction algorithm

Code used in tool `lbtt` (from `TCS`) to draw an LTL formula:

function RandomFormula(n):
if $n = 1$ **then**
    p := random symbol in $AP \cup \{\top, \bot\}$;
    return p;
else if $n = 2$ **then**
    op := random unary operator in $\{\neg, \mathbf{X}, \Box, \Diamond\}$;
    f := RandomFormula(1);
    return $op\ f$;
else
    op := random operator in $\{\neg, \mathbf{X}, \Box, \Diamond, \wedge, \vee, \rightarrow, \leftrightarrow, \mathbf{U}, \mathbf{R}\}$;
    if op *in* $\{\neg, \mathbf{X}, \Box, \Diamond\}$ **then**
        f := RandomFormula($n - 1$);
        return $op\ f$;
    else
        x := uniform integer in $[1, n-2]$;
        f_1 := RandomFormula(x);
        f_2 := RandomFormula($n - x - 1$);
        return ($f_1\ op\ f_2$);

Example: BST-like distribution

Consider the regular expressions $(+, \bullet, \star)$ on two letters a, b

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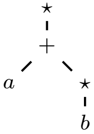
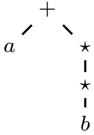


(ii)



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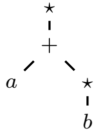

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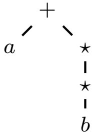

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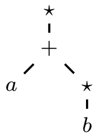

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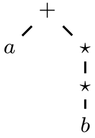

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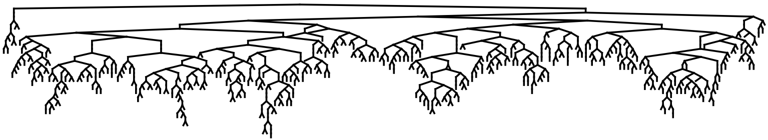
$\Longrightarrow$ Distribution not uniform for any choice of parameters.

BST-like trees: distribution over unary-binary trees



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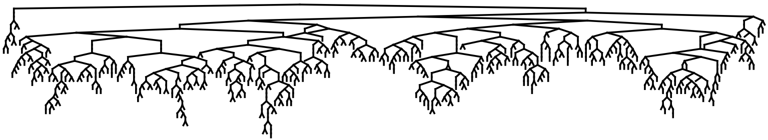


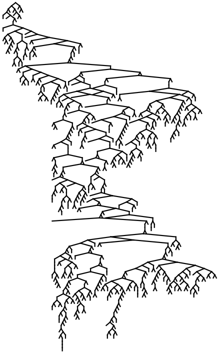
BST-like tree distribution is not uniform[a].

- Binary nodes ≈ balanced $\frac{n}{2}$–$\frac{n}{2}$,
  but for uniform trees

$$\mathbb{E}_n[\min(|T_L|, |T_R|)] \sim c_0\sqrt{n}\,.$$

- Expected height of different order

$$\Theta(\log n) \text{ vs } \Theta(\sqrt{n})\,.$$



[a]Tree T chosen uniformly from $\{T : |T| = n\}$.

Uniform and BST-like distributions

The uniform distribution:

- naturally maximizes entropy.
- can be sampled efficiently with some effort (Recursive method, Boltzmann samplers, Devroye's constrained GW).
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We had previously studied *semantically* **uniform expressions**...

Section

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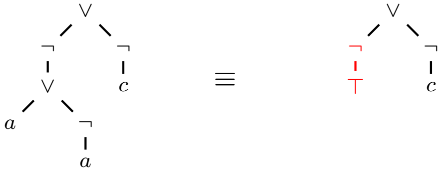

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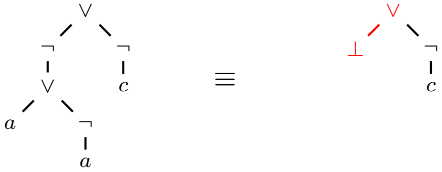

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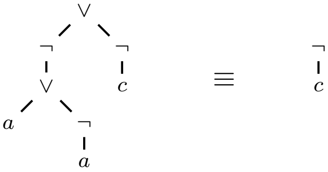

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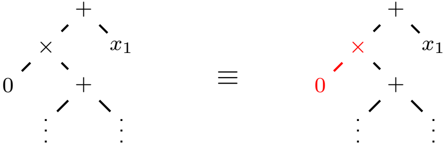

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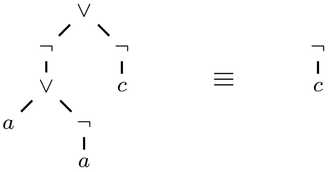

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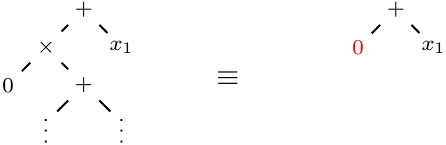

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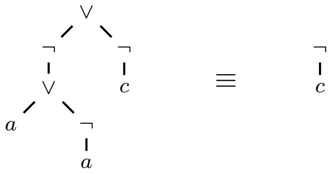

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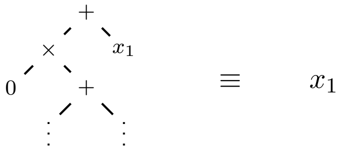

Question

Do semantic reductions affect the size of the random expressions?

Semantic simplification

Universal result for uniform tree model:

Theorem (Koechlin,Nicaud,R,'20)

Expected size of reduction of uniform tree bounded, as size $\rightarrow \infty$.

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- For regular expressions on two letters, $\texttt{constant bound} \approx 77.8$.

Absorbing patters: simplifying the trees

Definition (Simplification, absorbing pattern)

To define an absorbing pattern we fix:

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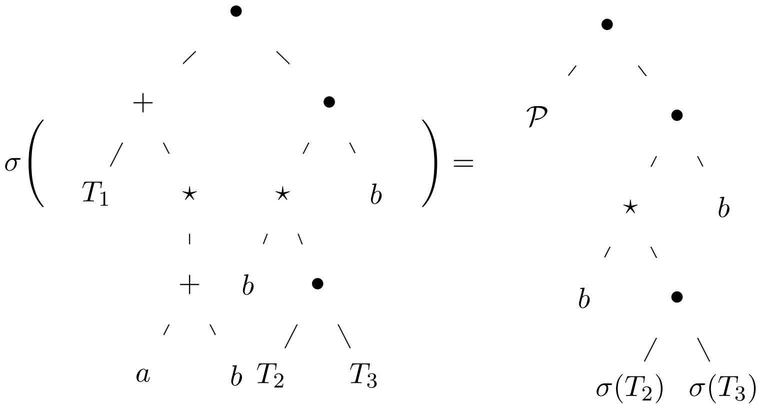

If we draw a random BST-like tree expression of size n:

- do we have the same flaw as uniform trees?

[2] Left to right $(p_\star, p_\bullet, p_+) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$,$(p_\star, p_\bullet, p_+) = (\frac{5}{29}, \frac{5}{29}, \frac{19}{29})$,$(p_\star, p_\bullet, p_+) = (\frac{1}{10}, \frac{1}{10}, \frac{8}{10})$

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Experimental expected size (10 000 samples)[2] on regular expressions $(+, \bullet, \star)$ on two letters a, b:

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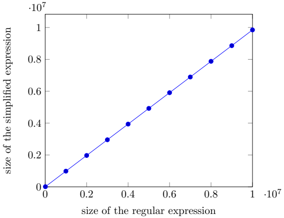

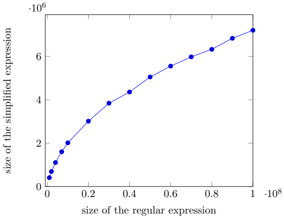

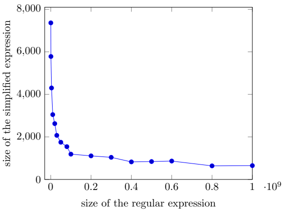

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1. Random BST-like tree model

2. Semantic simplifications

3. Result for BST-like trees and elements of the proof

4. Conclusions

Theorem. *Consider a family of expression trees defined from unary and binary operators with an absorbing pattern* $\mathcal{P}$ *for an operator* $\circledast$ *of arity* 2*.*

Take the simplification consisting in inductively changing a $\circledast$*-node by* $\mathcal{P}$ *whenever one of its children simplifies to* $\mathcal{P}$*.*

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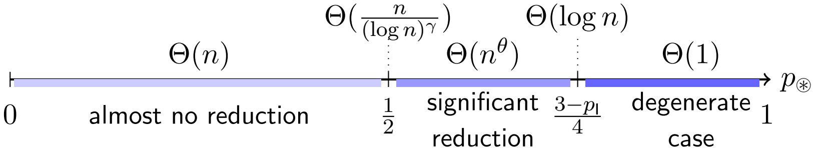

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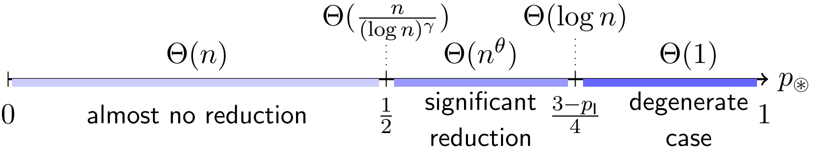
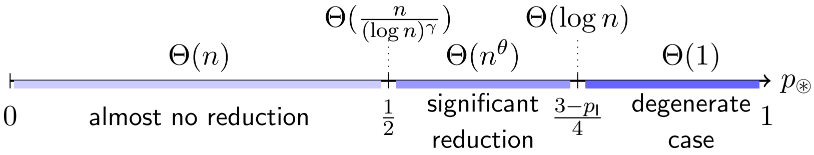


- Probability $p_\circledast$ of $\circledast$, and p_{l} of picking unary operator.
- Two critical points $p_\circledast = 1/2$ and $p_\circledast = (3 - p_{\mathsf{l}})/4$
- Regimes from no reduction $\Theta(n)$ to complete reduction $\Theta(1)$

The main regimes experimentally



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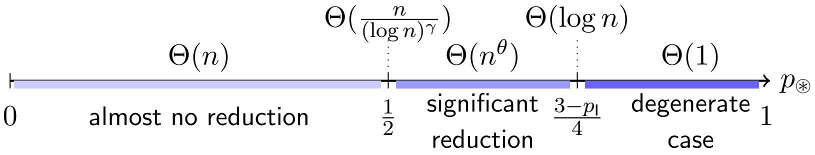

Experimental plots (10 000 samples)for regular expressions on two letters a, b: $\mathcal{P} = (a+b)^\star$ absorbing for union $\circledast = +$

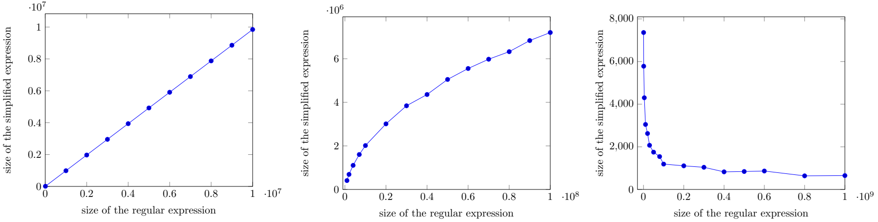

Figure: Left to right: linear ($p_+ = p_\star = p_\cdot = \frac{1}{3}$), sublinear ($p_+ = \frac{19}{29}$, $p_\star = p_\cdot = \frac{5}{29}$) and constant ($p_+ = \frac{8}{10}$, $p_\star = p_\cdot = \frac{1}{10}$).

Scheme of the proof: steps from Analytic combinatorics

We employ Analytic Combinatorics to study the expectation,

- Ordinary generating function

$$E(z) := \sum_{n=0}^{\infty} e_n z^n \,,$$

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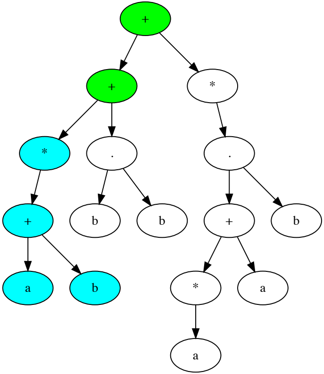
- **Analytic Step.** We look at $E(z)$ over the complex $z \in \mathbb{C}$.

  A *Transfer Theorem* links the behaviour of $E(z)$ at its dominant singularity to asymptotics of $e_n \Rightarrow$ Study singularities

$$E(z) \sim_{z \to 1} \lambda(1-z)^{-\alpha} \Longrightarrow e_n \sim \lambda n^{\alpha-1}/\Gamma(\alpha)$$

Symbolic step: fully reducible trees

An expression tree T is *fully reducible* when $\sigma(T) = \mathcal{P}$.



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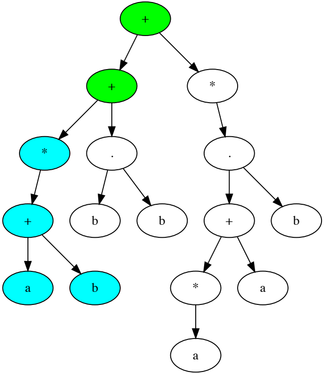

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- dictate the reduction process:
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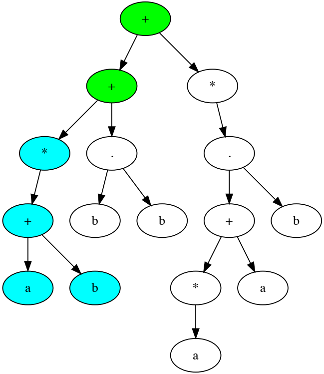

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$$\mathcal{P} = \underset{a \qquad b}{\overset{\star}{+}} \;; \quad \mathcal{R} = \mathcal{P} + \underset{\mathcal{R} \quad \mathcal{L}}{\overset{+}{/\ \backslash}} + \underset{\mathcal{L} \quad \mathcal{R}}{\overset{+}{/\ \backslash}} \,.$$

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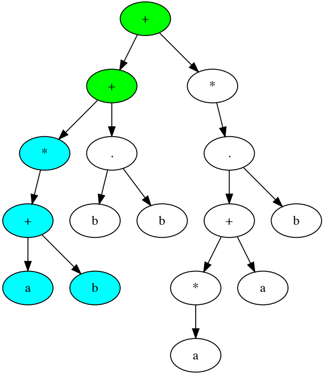

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We consider a fundamental sequence

$$\gamma_n := \Pr_n \{\sigma(T) = \mathcal{P}\} \, , \qquad A(z) := \sum \gamma_n z^n \, ,$$

of probabilities of *full* reduction and their generating function.

Symbolic step: recurrence for the expected value

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Recurrence for expected values

The recurrence for e_n involves γ_n,

$$e_{n+1} = 1 + (s-1)\gamma_{n+1}\mathbf{1}_{n+1 \neq s} + p_{\mathsf{I}} e_n + \frac{2p_{\mathsf{II}}}{n-1}\sum_{j=1}^{n-1} e_j + \frac{2p_{\circledast}}{n-1}\sum_{j=1}^{n-1}(e_j - s\gamma_j)(1-\gamma_{n-j}) ,$$

here $p_{\mathsf{II}} := 1 - p_{\mathsf{I}} - p_{\circledast}$ and $s = |\mathcal{P}|$.

Symbolic step: recurrence = differential equation

Recurrence yields first order differential equation

$$E'(z) = F(z, A(z)) + \tfrac{1}{1-p_{\mathsf{I}} z}\Big(\tfrac{2}{z} - p_{\mathsf{I}} + 2\,(1-p_{\mathsf{I}})\,\frac{z}{1-z} - 2p_{\circledast} A(z)\Big)\cdot E(z)\,,$$

in terms of $A(z) = \sum_n \gamma_n z^n$.

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Proof.

Differentiating we have $E'(z) = \sum (n+1) e_n z^n$. $\square$

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First order differential equations can be solved explicitly

Proposition

The equation $U'(z) = f(z) + g(z)U(z)$ where f, g are analytic functions on Ω has a unique solution analytic on Ω, satisfying $U(0) = u_0$,

$$U(z) = \exp\left(\int_0^z g(\zeta) d\zeta\right)\left(u_0 + \int_0^z f(\zeta)\exp\left(-\int_0^\zeta g(w)dw\right) d\zeta\right).$$

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Our coefficients depend on z and the unkown generating function $A(z)$.

Analytic step: the singularity $z = 1$

The generating functions $A(z)$ and $E(z)$

- are analytic for $|z| < 1$, as the series converge absolutely.

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Solution of ODE gives asymptotics

$$E(z) \sim \frac{c}{(1-z)^2} \left(2 + \int_0^z F(w, A(w)) I(w) dw \right) (I(z))^{-1}, \quad z \to 1,$$

where $I(z) := \exp\left(2p_{\circledast} \int_0^z \frac{A(w)}{1-p_{|}w} dw \right)$.

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To apply the Transfer Theorem and complete the proof:

- we require precise asymptotics for $A(z)$ at $z = 1$,
- we show that $A(z)$ and $E(z)$ are analytic over $\Omega = \mathbb{C} \setminus [1, \infty)$.

Fully reducible trees: probabilities

We study the generating function $A(z) = \sum \gamma_n z^n$

Proposition: recurrence for γ_n

The probabilities $\gamma_n = \Pr_n \{\sigma(T) = \mathcal{P}\}$ satisfy, for $n \geq |\mathcal{P}|$,

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Recurrence translates into Riccati differential equation

$$A'(z) = (s-2)\gamma_s z^{s-1} + \left(\frac{2}{z} + 2p_{\circledast} \frac{z}{1-z} \right) A(z) - p_{\circledast} \cdot (A(z))^2 \,,$$

where $s = |\mathcal{P}|$ is the size of the absorbing pattern.

Behaviour of solutions of Riccati ODE

Considering $v(z)$ such that $p_{\circledast} A(z) = v'(z)/v(z)$,

Riccati equation becomes linear

$$v''(z) = p_{\circledast} \cdot (s-2)\gamma_s z^{s-1} v(z) + \left(\frac{2}{z} + 2p_{\circledast} \frac{z}{1-z} \right) v'(z) \,.$$

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Proposition

The generating function $A(z)$ satisfies, $z \to 1$

- For $p_\circledast > \frac{1}{2}$, $A(z) = \frac{\gamma_\infty}{1-z} + O((1-z)^{2p_\circledast - 2})$,
- For $p_\circledast = \frac{1}{2}$, $A(z) = \frac{2}{1-z} \left(\log\left(\frac{1}{1-z}\right)\right)^{-1} \left(1 + O\left(\log\left(\frac{1}{1-z}\right)^{-1}\right)\right)$
- For $p_\circledast < \frac{1}{2}$, $A(z) \sim \frac{D}{(1-z)^{2p_\circledast}}$,

where $\gamma_\infty := (2p_\circledast - 1)/p_\circledast$ and $D > 0$ is a constant.

Probability of full reduction

Theorem

The probability γ_n of being fully reducible tends to the constant $\gamma_\infty := (2p_\circledast - 1)/p_\circledast$ for $p_\circledast > \frac{1}{2}$ and to zero otherwise.

Moreover,

- *for $p_\circledast = \frac{1}{2}$ we have $\gamma_n \sim \frac{2}{\log n}$,*
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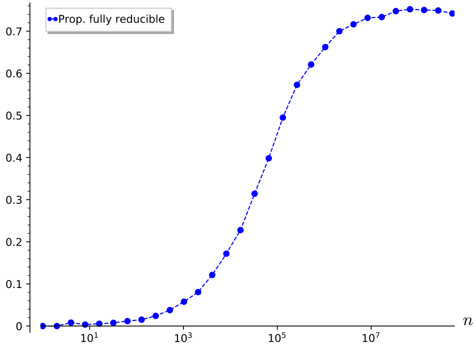
Experimental plot:
regular expressions on two letters with

$$(p_+, p_\bullet, p_\star) = \left(\frac{8}{10}, \frac{1}{10}, \frac{1}{10}\right).$$

Then

$$\lim_{n\to\infty} \gamma_n = 3/4\,.$$



Proof principles: analytic step – Frobenius method

Recall. From the Riccati equation, we have a 2nd order linear ODE

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and we want the behavior around $z = 1$ (false pole at $z = 0$).

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Let $v''(z) = \frac{q(z)}{(1-z)^2} v(z) + \frac{p(z)}{1-z} v'(z)$ with $q(z)$ and $p(z)$ analytic at $z = 1$.

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If α is a root of the *indicial equation*

$$X^2 = (p(1) + 1)X + q(1)\,,$$

there is $h_\alpha(z)$ analytic at 1, $h_\alpha(1) = 1$, such that $(1-z)^\alpha h_\alpha(z)$ is a solution of the ODE.

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- If $\alpha_1 - \alpha_2 \notin \mathbb{Z}$, solutions are linearly independent.
- If $\alpha_1 - \alpha_2 \in \mathbb{Z}$, factor $(1-z)^{|\alpha_1 - \alpha_2|}$ is polynomial.
  $\implies$ to obtain independent solution multiply $\times \log(1-z)$.

Section

Conclusions and further work

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   $\Rightarrow$ expect similar results, threshold $\frac{1}{a}$ instead of $\frac{1}{2}$.
2. Absorbing pattern model is general
   $\Rightarrow$ consider interactions between operators?

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Questions and further work

1. Absorbing operator $\circledast$ of arity $a \geq 3$?
   $\Rightarrow$ expect similar results, threshold $\frac{1}{a}$ instead of $\frac{1}{2}$.
2. Absorbing pattern model is general
   $\Rightarrow$ consider interactions between operators?
3. Take a concrete case: LTL formulas.

References

Philippe Flajolet and Robert Sedgewick.
Analytic Combinatorics.
Cambridge University Press, Cambridge (2009).
https://algo.inria.fr/flajolet/Publications/AnaCombi/anacombi.html

Florent Koechlin, Cyril Nicaud, and Pablo Rotondo.
On the Degeneracy of Random Expressions Specified by Systems of Combinatorial Equations.
Proceedings of DLT 2020, LNCS vol. 12086, pp 164–177.
https://doi.org/10.1007/978-3-030-48516-0_13

Florent Koechlin and Pablo Rotondo.
Analysis of an efficient reduction algorithm for random regular expressions based on universality detection.
Proceedings of CSR 2021. LNCS, vol 12730. Springer.
https://doi.org/10.1007/978-3-030-79416-3_12

Florent Koechlin and Pablo Rotondo.
Absorbing patterns in BST-like expression-trees.
Proceedings of STACS 2021, LIPICs, vol.187, 48:1–48:15
https://doi.org/10.4230/LIPIcs.STACS.2021.48