A Probabilistic Model Revealing Shortcomings in Lua's Hybrid Tables

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Joint work with

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  - Engineers sometimes choose innovative implementations
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$\Rightarrow$ `Lua 5.0` introduced several innovations,
among them a new Table structure.

The Lua programming language[1]

❖ What is Lua?

Lua is a powerful, efficient, lightweight, embeddable scripting language. It supports procedural programming, object-oriented programming, functional programming, data-driven programming, and data description.

Lua combines simple procedural syntax with powerful data description constructs based on associative arrays and extensible semantics. Lua is dynamically typed, runs by interpreting bytecode with a register-based virtual machine, and has automatic memory management with incremental garbage collection, making it ideal for configuration, scripting, and rapid prototyping.

❖ Why choose Lua?

Lua is a proven, robust language

Lua has been used in many industrial applications (e.g., Adobe's Photoshop Lightroom), with an emphasis on embedded systems (e.g., the Ginga middleware for digital TV in Brazil) and games (e.g., World of Warcraft and Angry Birds). Lua is currently the leading scripting language in games. Lua has a solid reference manual and there are several books about it. Several versions of Lua have been released and used in real applications since its creation in 1993. Lua featured in HOPL III, the Third ACM SIGPLAN History of Programming Languages Conference, in 2007. Lua won the Front Line Award 2011 from the Game Developers Magazine.

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Lua is fast

Lua has a deserved reputation for performance. To claim to be "as fast as Lua" is an aspiration of other scripting languages. Several benchmarks show Lua as the fastest language in the realm of interpreted scripting languages. Lua is fast not only in fine-tuned benchmark programs, but in real life too. Substantial fractions of large applications have been written in Lua.

If you need even more speed, try LuaJIT, an independent implementation of Lua using a just-in-time compiler.

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> The implementation of tables in Lua involves some clever algorithms. Every table in Lua has two parts: the *array* part and the *hash* part. The array part stores entries with integer keys in the range 1 to n, for some particular n. (We will discuss how this n is computed in a moment.) All other entries (including integer keys outside that range) go to the hash part.

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- In our work we study this mechanism.

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- A problem for more realistic scenarios ?

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Consider $p > \frac{1}{2}$. A random sequence of T insertion/deletions:

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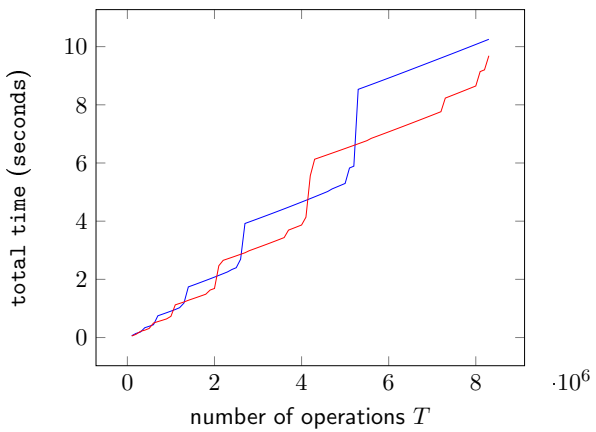

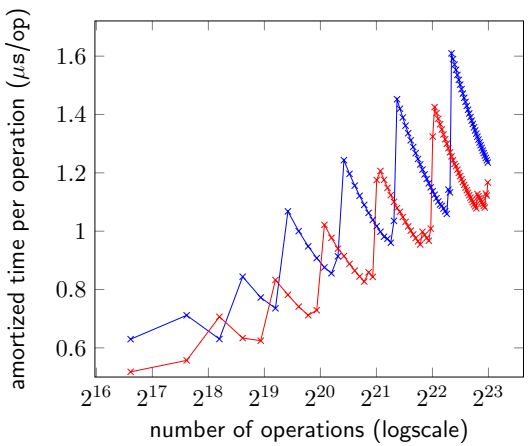

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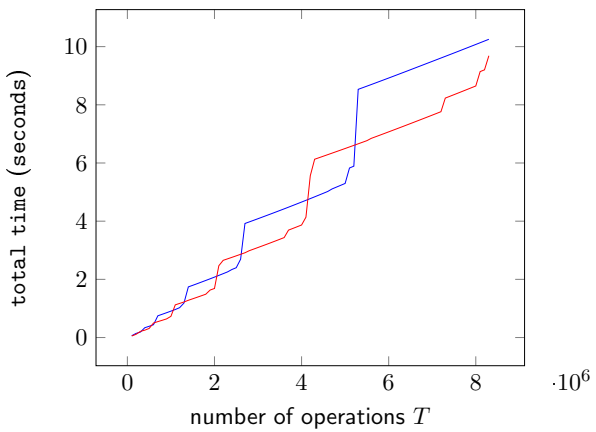

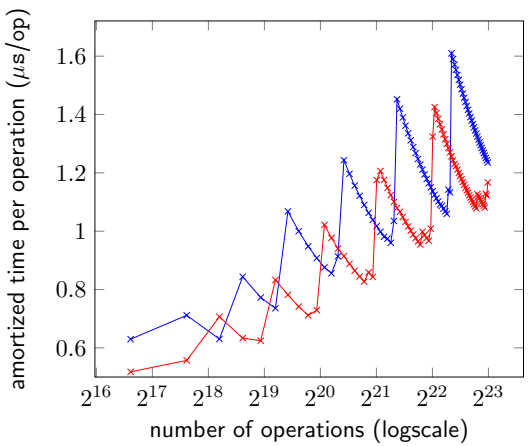

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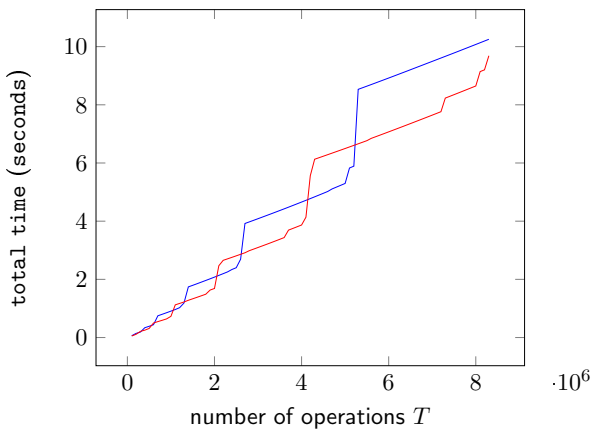

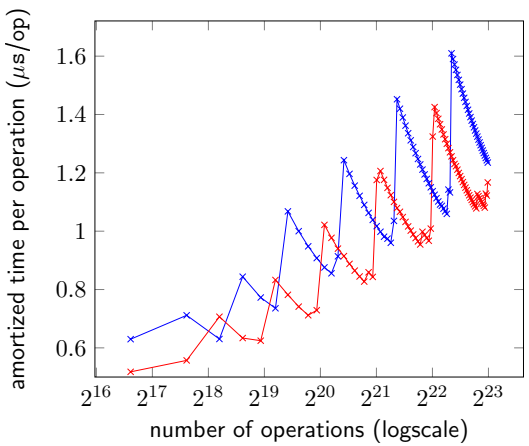

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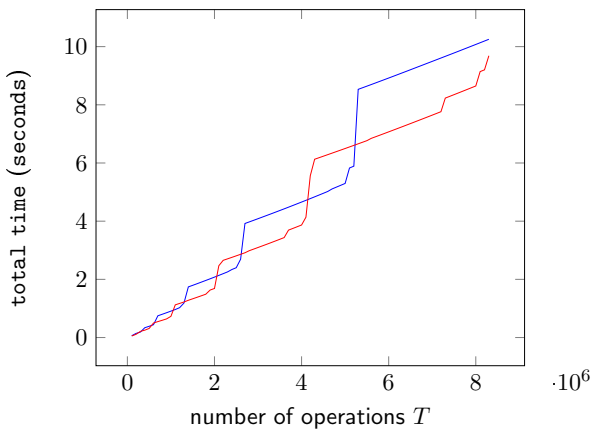

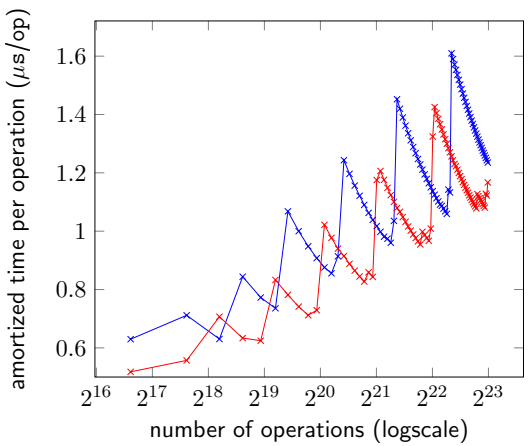

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We propose potential fixes... which we will see and implement during the talk

Plan of the talk

1. The Lua hashmap
2. The probabilistic model
3. Hybrid Tables and insertions
4. Conclusions and further work

A (very) short intro to hashing

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- array H of size M much smaller than $|\mathcal{K}|$,
- hash function $h: \mathcal{K} \to \mathbb{Z}_{\geq 0}$,
- store $x \in \mathcal{K}$ in slot $H[h(x) \bmod M]$.

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Hash-functions must

- avoid collisions as much as possible,
- be fast to compute.

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We still have to decide what to do in case of a collision:

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. . . rehash into larger array H'

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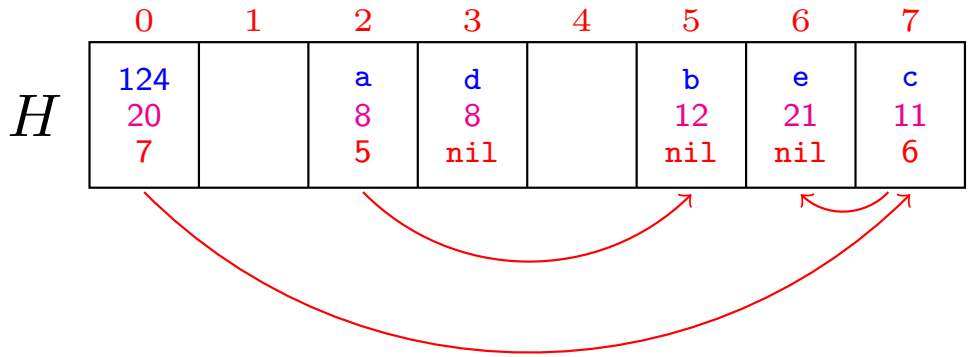

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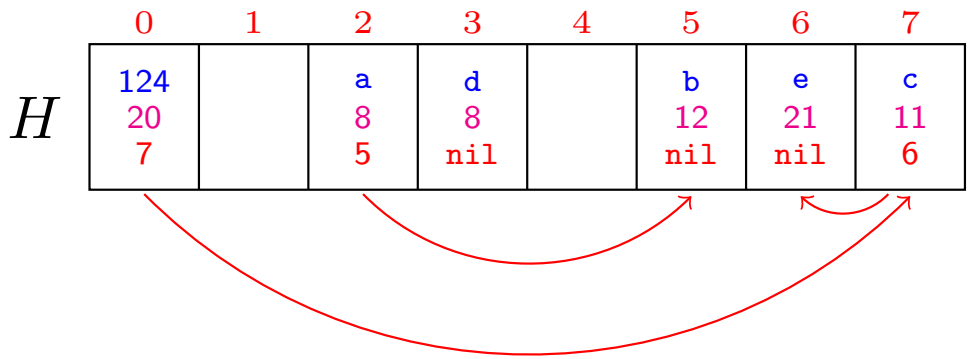

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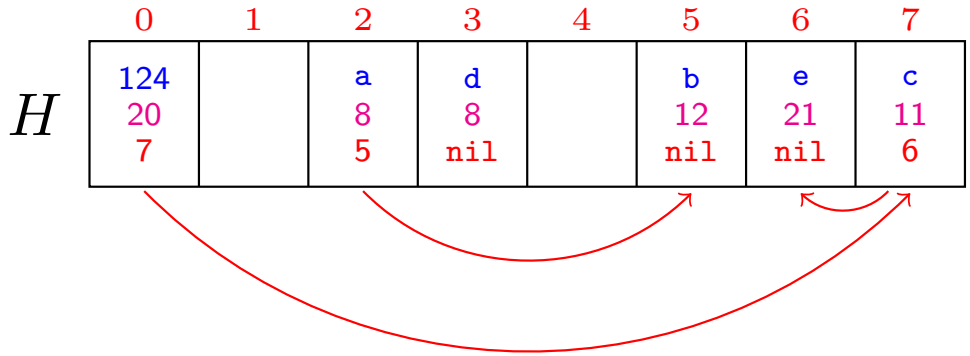

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when K is odd, and this will lead to a substantial bias in many files. It would be even worse to let M be a power of the radix of the computer, since K mod M would then be simply the least significant digits of K (independent of the other digits). Similarly we can argue that M probably shouldn't be a multiple of 3;

Figure: Extract from The Art of Computer Programming (vol. 3)

Hashmap mechanism: hash function

Not immediate that h may be a bad choice

- fast for integers: $x \bmod M = x\&(M-1)$,
- more involved for strings:

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unsigned int luaS_hash (const char *str, size_t l,
     unsigned int seed) {
  unsigned int h = seed ^ cast_uint(l);
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In this talk we do not discuss the choice of the hash-function h,

Base assumption

The function $x \mapsto h(x) \bmod M$ is roughly uniform

Hashmap mechanism: insertion

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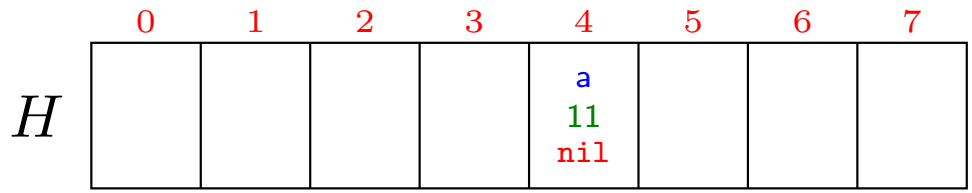

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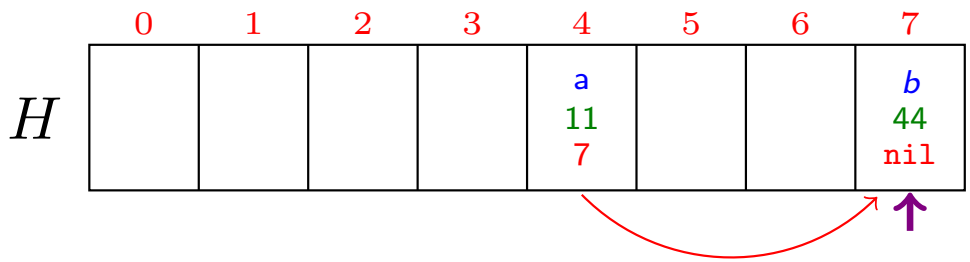

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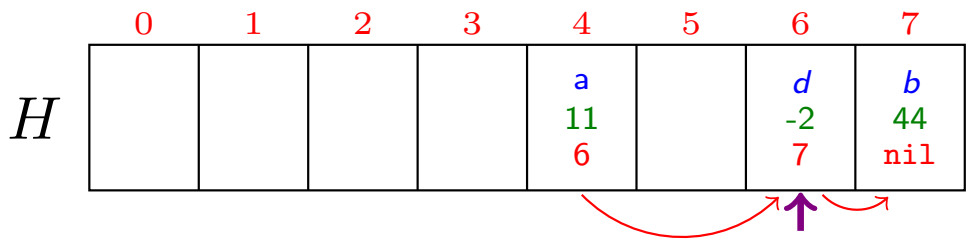

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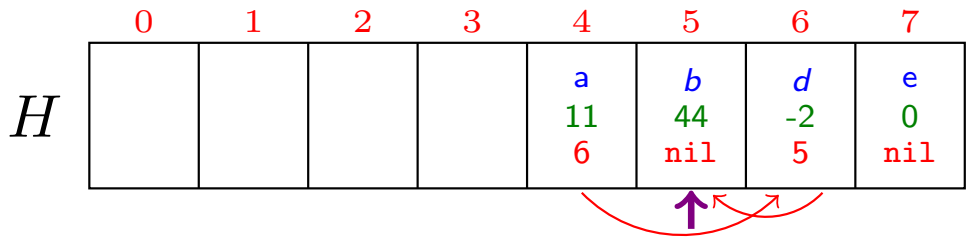

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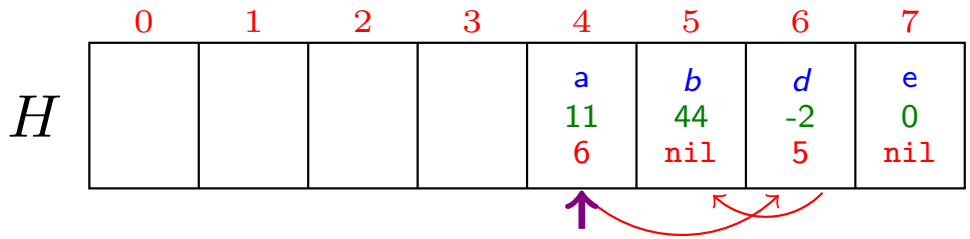

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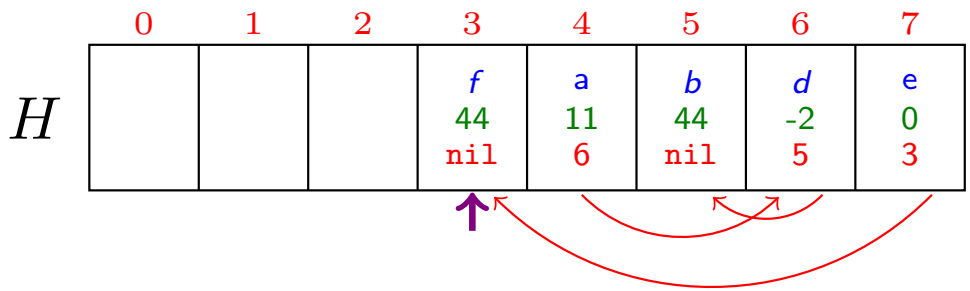

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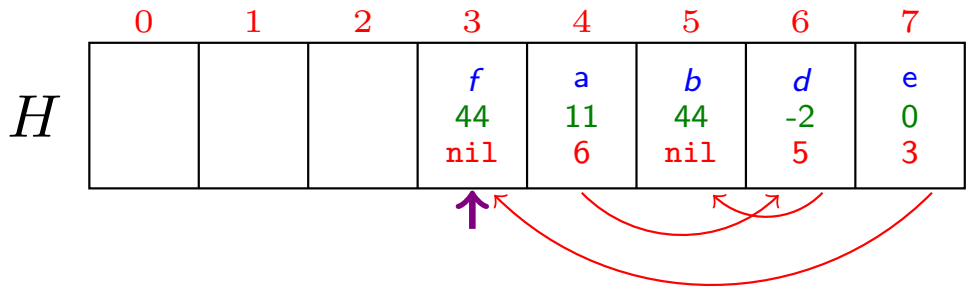

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Finding free position: pointer from right to left.

Hashmap mechanism: insertion

Insertions work as follows: we want to insert key x

- if position $h(x)$ free $\Rightarrow$ insert

  ... else position $h(x)$ is occupied by key y,

- if $h(y) = h(x) \Rightarrow$ put x into a free position, update chain

  $$\texttt{pos}(y) \to \texttt{pos}(z) \to \ldots \texttt{ to } \texttt{pos}(y) \to \texttt{pos}(x) \to \texttt{pos}(z) \to \ldots$$

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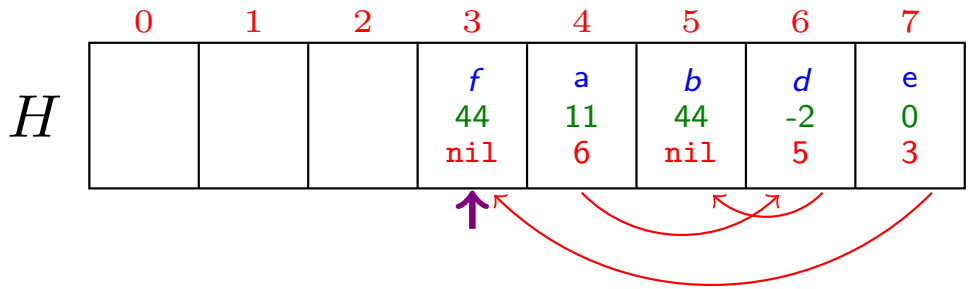

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. . . deleted spots are cleaned up during rehashing

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. . . but it is not very realistic, users do not behave this way (?)

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We set a more interesting yet simple model

Probabilistic model

Fix $p > \frac{1}{2}$ and apply T insertion/deletions from an empty table:

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- Each rehash costs linear time $\Theta(M)$. ☹

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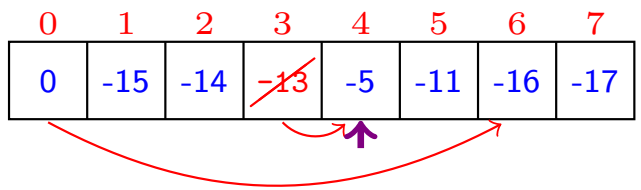

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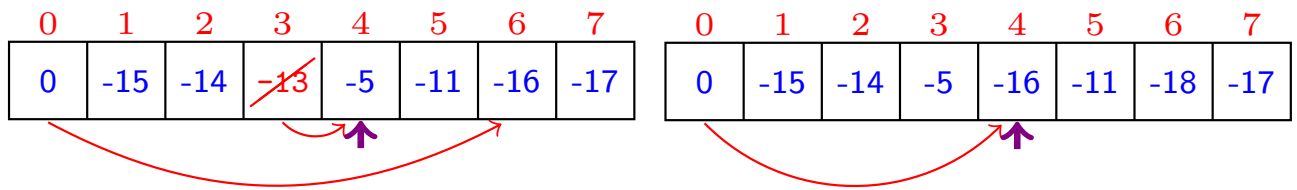

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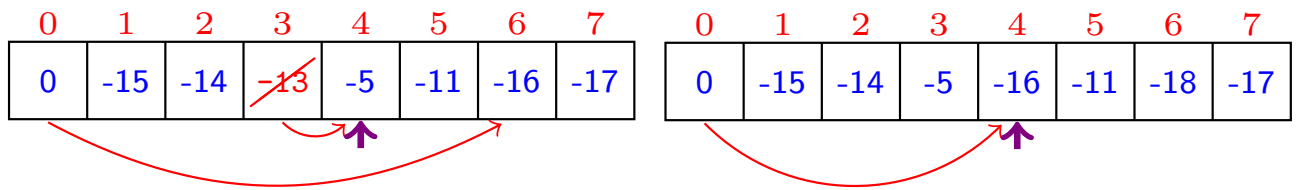

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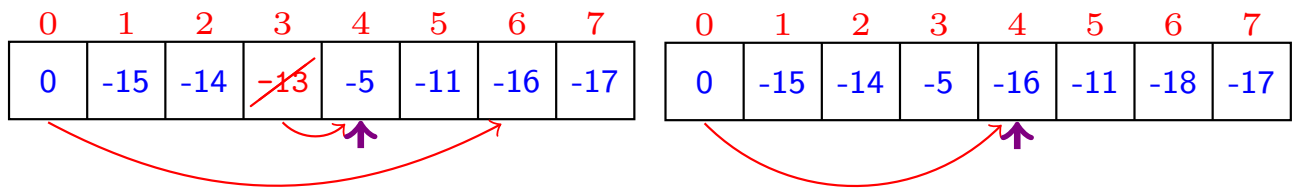

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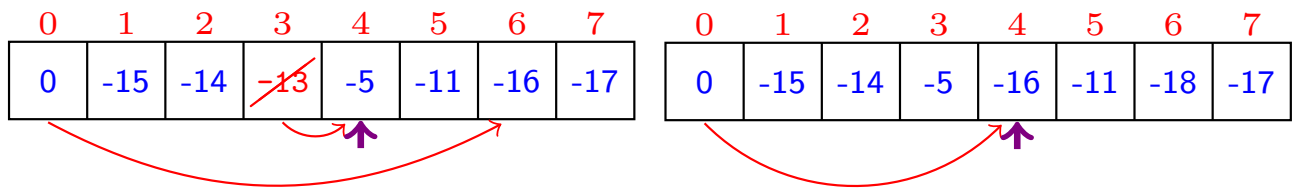

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... useless rehashes go on until we hit a rehash with $\delta = 0$

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With high probability, `Lua` uses $\Omega(T \log T)$ time for this process.

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... at least $\log M$ rehashes to increase M

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Under these *simplifying assumptions*, between two rehashes, the number of deleted cells satisfies the recurrence (starting from $\delta_{t_0} = 0$)

$$\delta_{t+1} = \begin{cases} \delta_t - 1 & \text{with probability } \frac{p\delta_t}{M} & [\textit{insertion at deleted key}], \\ \delta_t & \text{with probability } p\left(1 - \frac{\delta_t}{M}\right) & [\textit{insertion at free cell}], \\ \delta_t + 1 & \text{with probability } 1 - p & [\textit{deletion}]. \end{cases}$$

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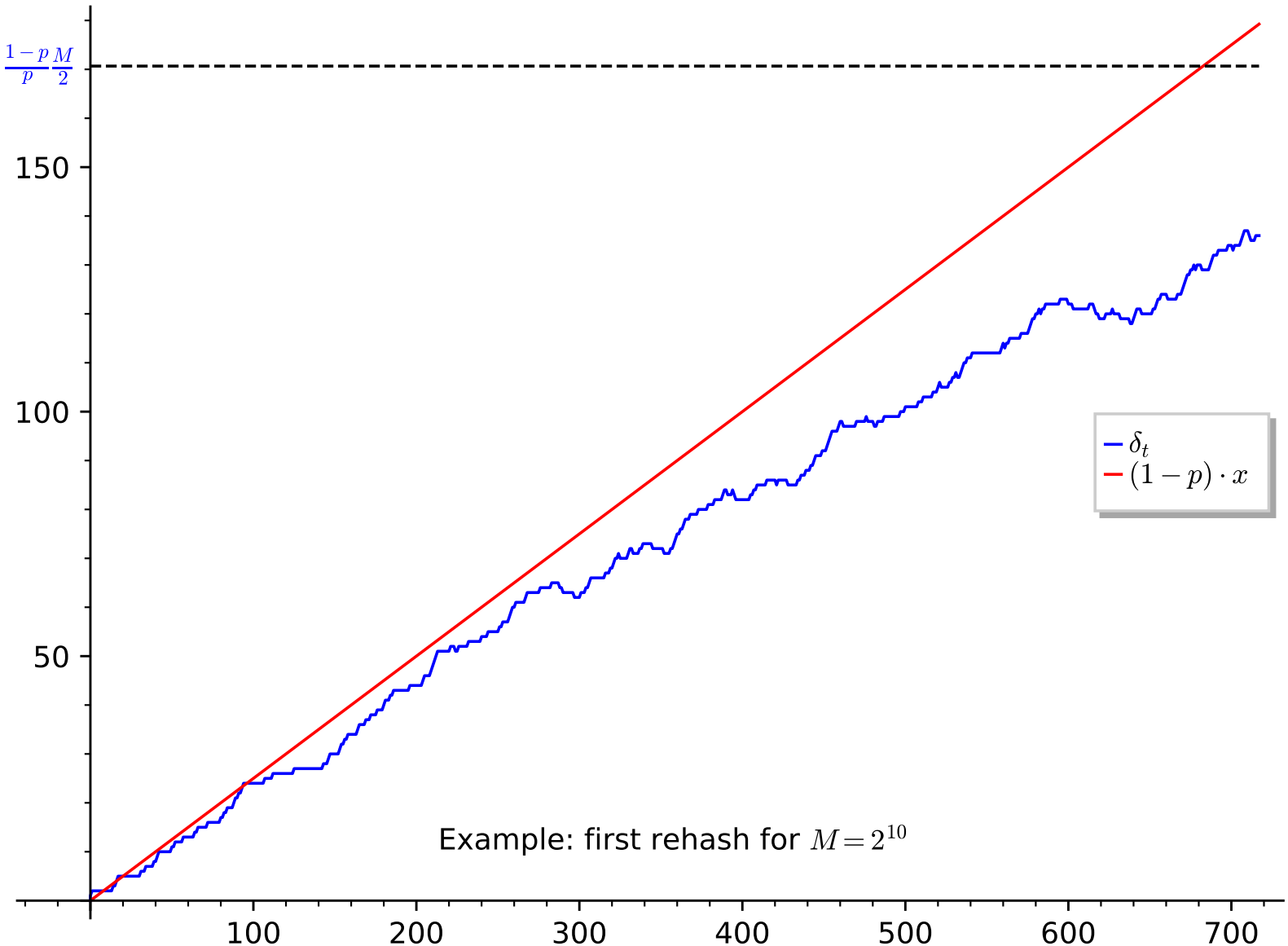
- Rehash occurs before reaching equilibrium
  - ... at the beginning δ_t increases linearly
  - ... as we approach equilibrium, increase weakens

Evolution of number of deleted cells δ_t: linear increase



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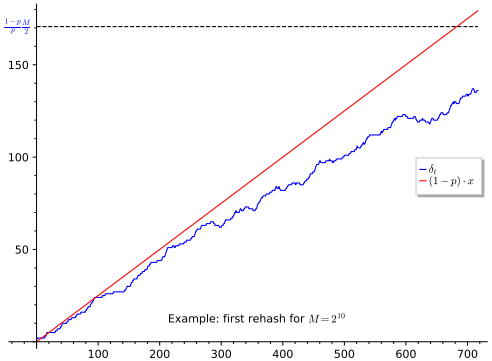

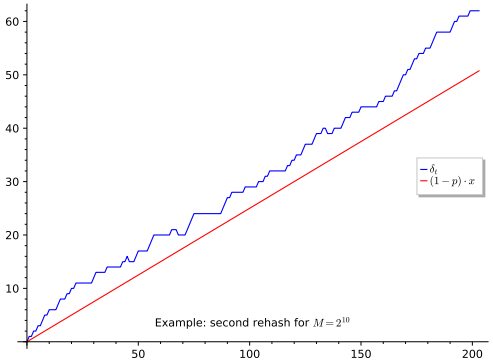

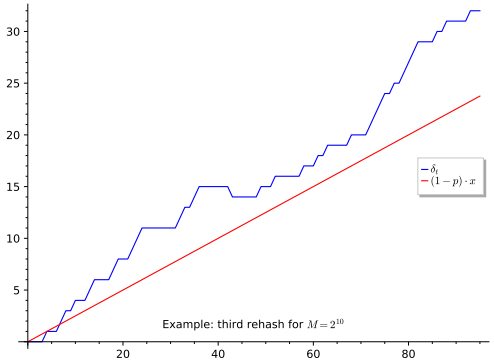

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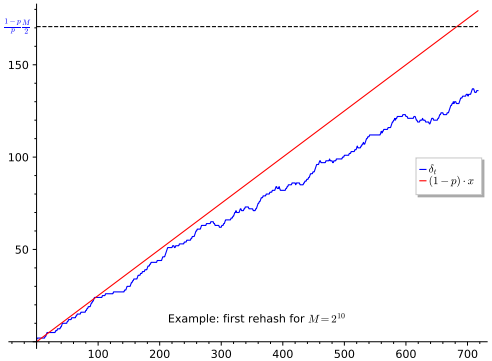

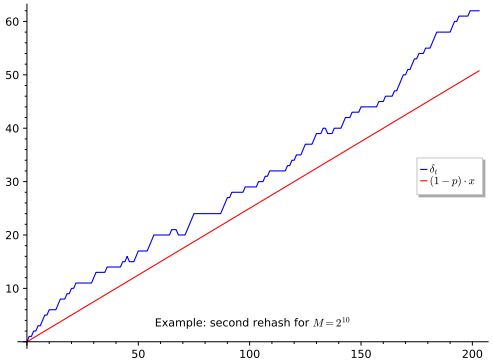

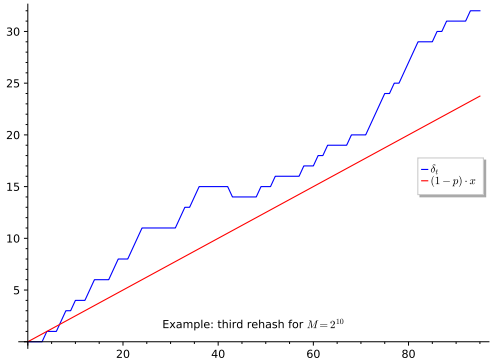

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Let $\texttt{free}_0$ free (unused) cells after last rehash (set $t = 0$)

Remark: there is a linear regime for a proportion of time

After $t = \left\lfloor \frac{1-p}{p} \texttt{free}_0 \right\rfloor < \texttt{free}_0$ steps

$$\frac{p\delta_t}{M} \leq (1-p)\frac{\texttt{free}_0}{M} \leq \frac{1-p}{2},$$

$\Delta\delta_t = 1$ still twice as likely as $\Delta\delta_t = -1$.

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- By time $t = \lfloor \frac{1-p}{p} \texttt{free}_0 \rfloor$, number of deleted cells δ_t increased linearly.
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With high probability equilibrium has never been reached by time τ.

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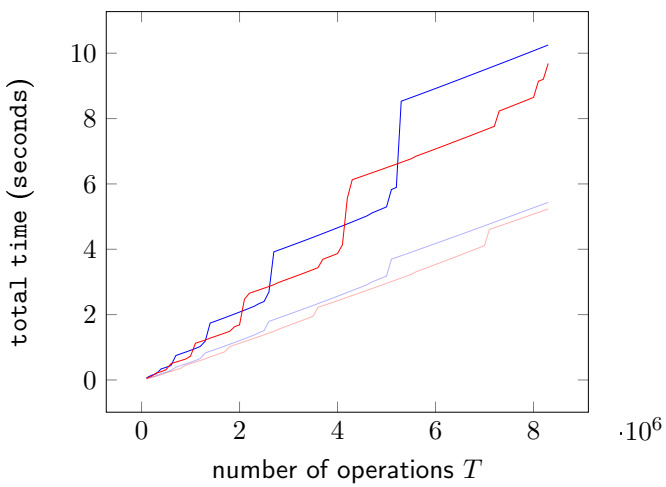

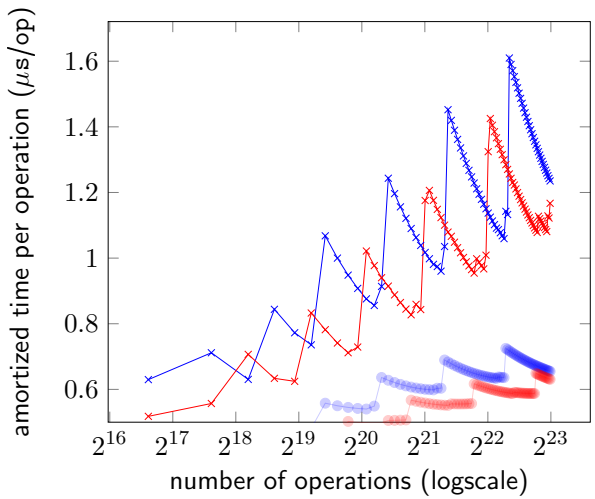

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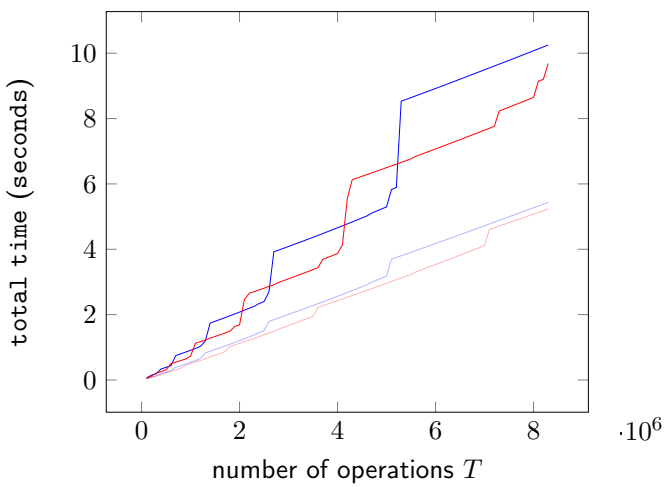
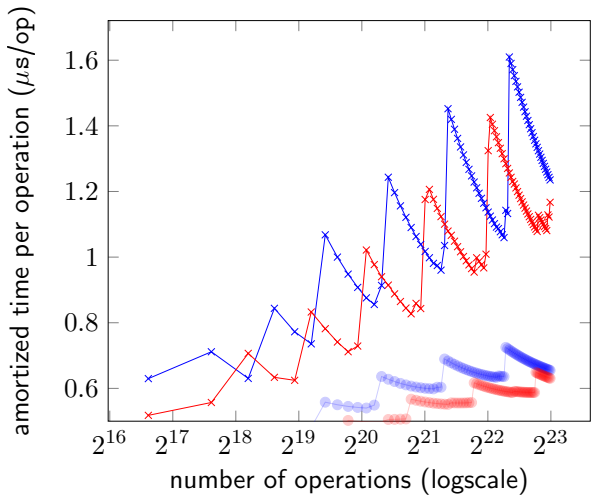




Ensuring a proportion $\beta \in (0, 1)$ of empty cells

- we require at least βM operations before rehashing again

Fixing the Lua hash table

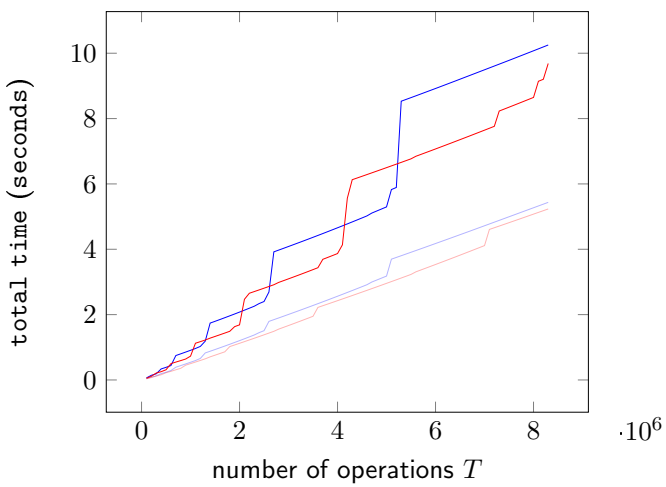
Potential solutions:

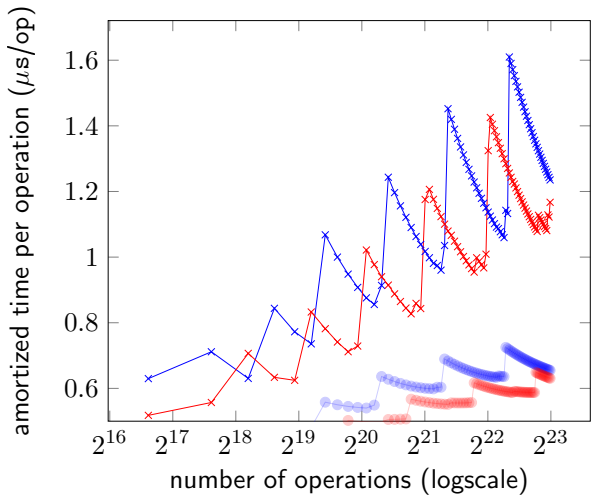
- implement real deletions

  . . . requires several changes (update chains, list of available cells)

- keep a minimum proportion of free cells after rehashing

  . . . we will implement it now in 2 minutes for ≥ 20%





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- we require at least βM operations before rehashing again
- amortized #insertions per operation $\leq (M + \beta M)/(\beta M) = 1 + \beta^{-1}$

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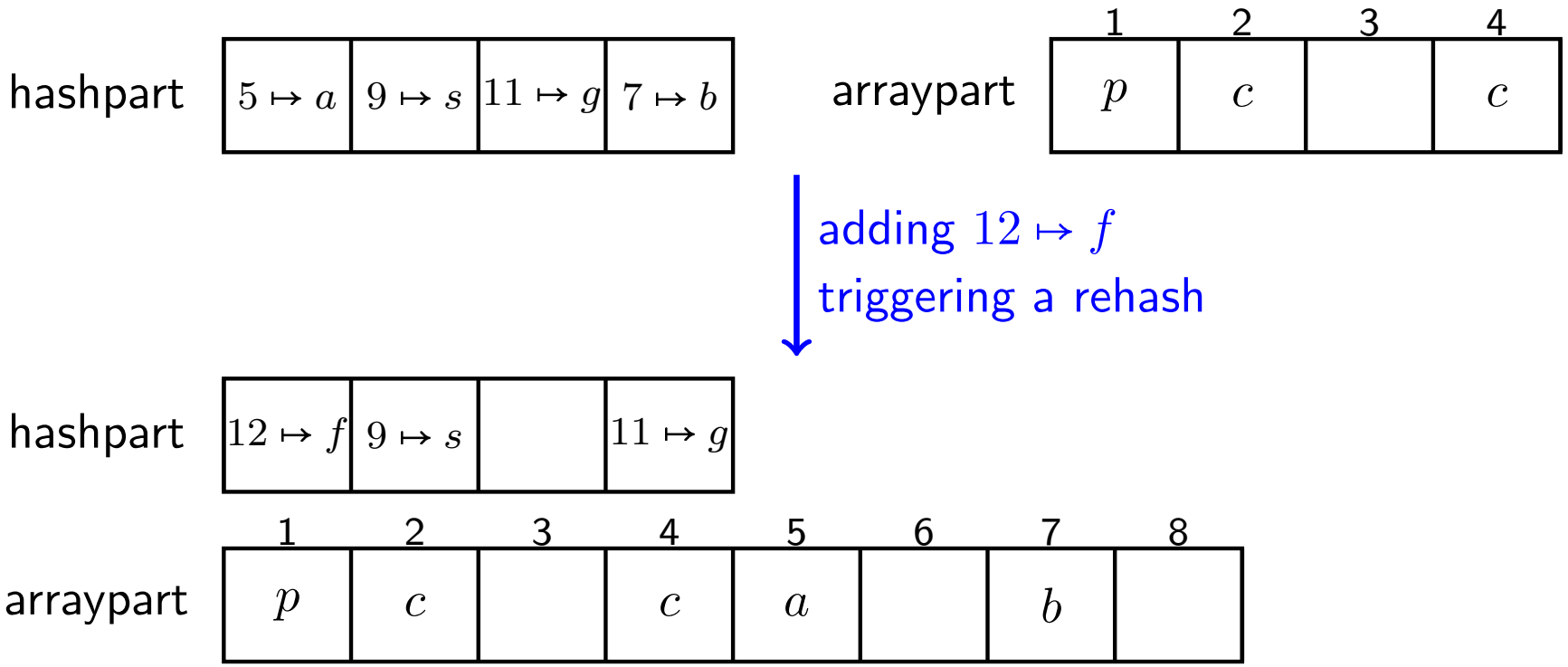

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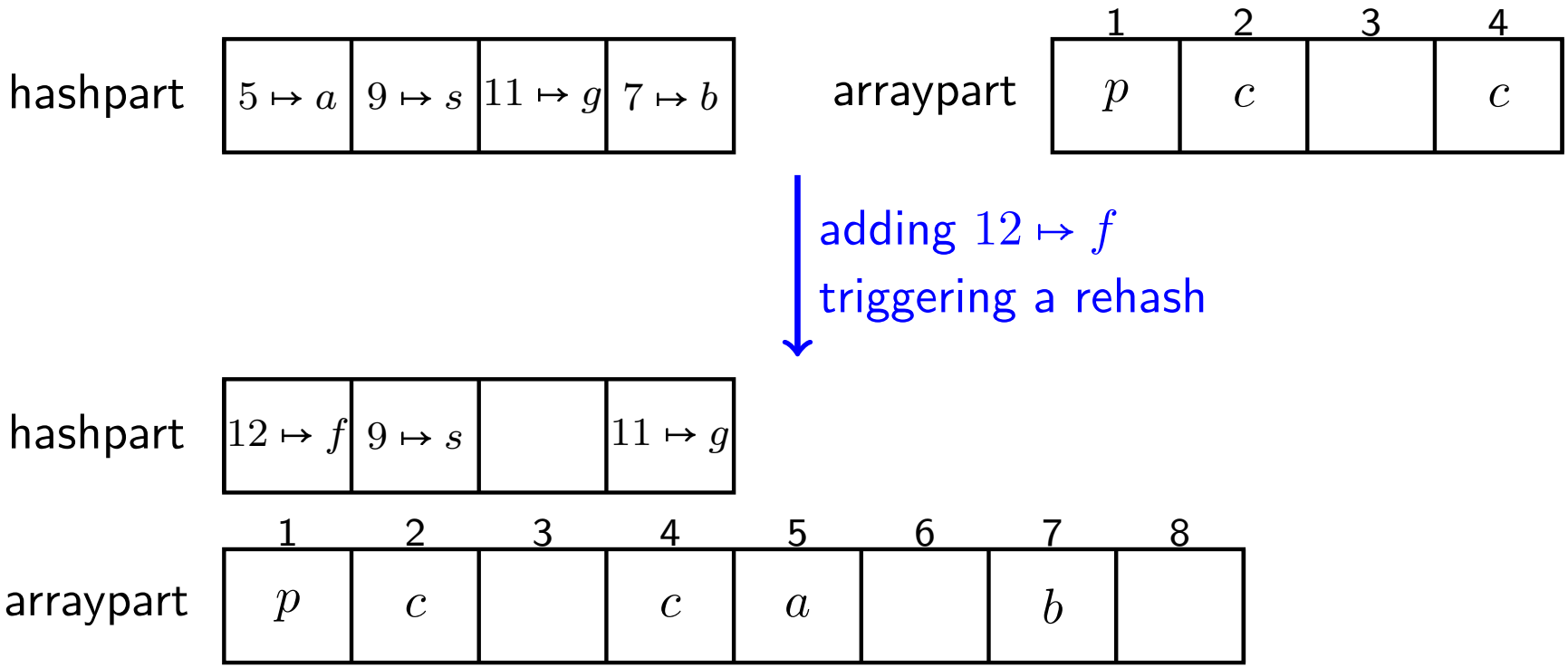

. . . rehash just *emulated* a deletion in the hash-part !

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- keys $2^j + 1, j \geq 0$, induce rehash, rest inserted in array-part directly

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$\Longrightarrow$ time $\Omega(k \cdot 2^k)$

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A permutation of $[1, \ldots, n]$

Permutations of $[1, \ldots, n]$ are a natural choice:

- they come up in practice in many contexts,
- they give the array-part a good shot at being used, yet ...

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... but this is rather unlikely for a permutation

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$\Longrightarrow$ array part not really used and complexity essentially linear

Theorem (Martínez,Nicaud,R 2022)

Inserting random permutation takes time $\mathcal{O}(n\, g(n))$ provided that n does not approximate powers of two from above[a].

[a] Fix $b \in (1,2)$, $n \in \mathbb{N}_b := \bigcup_{j \geq 0} \{k : 2^j b < k \leq 2^{j+1}\}$.

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Conclusions

- Will Lua conceptors take this into account?
- Important to model and study algorithms implemented in practice.

Thank you!