Convergence and the Harmonic series

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February 19, 2024

Proposition 1. *Suppose $q(k) > 0$ is a sequence such that $\sum_k q(k) < \infty$. Then, for every $\varepsilon > 0$, $D = D_\varepsilon = \{j : q(j) \geq \varepsilon/j\}$ has natural density 0, namely*

$$\lim_{n\to\infty} \tfrac{1}{n} \left|\{j \leq n : q(j) \geq \varepsilon/j\}\right| = 0\,.$$

Proof. Suppose otherwise. Then there is a sub-sequence (n_k) of the positive integers such that $\frac{1}{n_k}\left|\{j \leq n_k : q(j) \geq \varepsilon/j\}\right| \to \delta$ for some $\delta > 0$. By taking further a sub-sequence if necessary, we may assume without loss of generality that $n_k/n_{k+1} \to 0$. Let us remark then that this implies $\frac{1}{n_{k+1}}\left|\{n_k < j \leq n_{k+1} : q(j) \geq \varepsilon/j\}\right| \to \delta$.

Fix any $\delta' > 0$ with $\delta' < \delta$. Then, for all large enough $k \geq K$ we have

$$\tfrac{1}{n_{k+1}}\left|\{n_k < j \leq n_{k+1} : q(j) \geq \epsilon/j\}\right| > \delta'\,.$$

Let us decompose the partial sums of $\sum q(j) < \infty$ as follows

$$\sum_{j=1}^{n_k} q(j) = \sum_{j=1}^{n_1} q(j) + \sum_{i=1}^{k-1} \sum_{j=n_i+1}^{n_{i+1}} q(j)\,.$$

Let us note that here, for $i \geq K$,

$$\sum_{j=n_i+1}^{n_{i+1}} q(j) \geq \sum_{n_i < j \leq n_{i+1} : q(j) \geq \varepsilon/j} (\varepsilon/j) \geq \tfrac{\varepsilon}{n_{i+1}}\left|\{n_i < j \leq n_{i+1} : q(j) \geq \varepsilon/j\}\right| > \varepsilon\,\delta'\,.$$

Therefore we deduce that, for $k \geq K$,

$$\sum_{j=1}^{n_k} q(j) \geq (k-K)\,\varepsilon\,\delta' \to \infty\,,$$

as $k \to \infty$, a contradiction to the convergence of the series $\sum q(j)$. □

This means that if you pick j uniformly at random from $\{1,\ldots,n\}$, the probability of having $jq(j) \geq \varepsilon$ gets smaller and smaller as $n \to \infty$. This can be made precise as follows. Consider U_n a uniform real number from $[0,1]$, and let $X_n := \lceil nU_n \rceil q(\lceil nU_n \rceil)$. Then $X_n \to 0$ in probability. Note that U_n need not depend on n.