Analytic Combinatorics of Unlabeled Objects

Set of exercises IV: responses

1. **The generating function of the central coefficient**

In this exercise we use complex-integration to prove that

$$\sum_{n=0}^{\infty} \binom{2n}{n} z^n = \frac{1}{\sqrt{1-4z}}\,,$$

for $|z| < 1/4$.

(a) Prove that $F(z) \triangleq \sum_{n=0}^{\infty} \binom{2n}{n} z^n$ converges for $|z| < \frac{1}{4}$.

(b) Fix $r > 0$. Remark that $\oint_{|u|=r} \frac{(1+u)^{2n}}{u^{n+1}} du = 2\pi \binom{2n}{n}$. Use this to prove $F(z) = \frac{1}{\sqrt{1-4z}}$.

(c) Deduce that $\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{n+1} z^n = \frac{1-\sqrt{1-4z}}{2z}$.

Responses.

(a) We note that $\binom{2n}{n} \leq \sum_j \binom{2n}{j} = 4^n$, thus the coefficients are dominated by those of $|4^n z^n|$ and the series converges absolutely for $|z| < \frac{1}{4}$.

(b) The first identity follows from the Binomial Theorem, $[u^n](1+u)^{2n} = \binom{2n}{n}$ and Cauchy's Formula applied for the circle (the simplest case, which could be computed directly). Consider $|z| < 1/4$ and let $r = 1$. Then $\frac{(1+r)^2}{r}|z| = 4|z| < 1$. Thus $|\frac{(1+u)^2}{u} z| \leq \frac{(1+r)^2}{r}|z| < 1$ and the series converges $\sum \left(\frac{(1+u)^2}{u} z\right)^n$ converges uniformly. Integrating over $|u| = r = 1$, by the uniform convergence we may exchange sums and integrals:

$$\begin{aligned}
\sum_{n=0}^{\infty} \binom{2n}{n} z^n &= \tfrac{1}{2\pi i} \sum_{n=0}^{\infty} \left(\oint_{|u|=1} \frac{(1+u)^{2n}}{u^{n+1}} du \right) z^n \\
&= \tfrac{1}{2\pi i} \oint_{|u|=1} \left(\sum_{n=0}^{\infty} \left(\tfrac{(1+u)^2}{u} z \right)^n \right) \tfrac{du}{u} \\
&= \tfrac{1}{2\pi i} \oint_{|u|=1} \frac{1}{1 - \frac{(1+u)^2}{u} z} \frac{du}{u}\,.
\end{aligned}$$

Now it all comes down to compute the latter integral. Cleaning out the denominators, we want to integrate $u \mapsto \frac{1}{u-(1+u)^2 z}$ over the circle $|u| = 1$. The denominator has two potential roots:

$$u_{\pm} = u_{\pm}(z) = \frac{1 - 2z \pm \sqrt{1-4z}}{2z}\,.$$

We remark, by the Newton relations for the roots of the polynomials, that $u_+ \times u_- = 1$, because these are roots of $u^2 + (2 - 1/z)u + 1 = 0$. Hence it follows that $|u_+| \times |u_-| = 1$. Moreover $u_+ + u_- = 1/z - 2$, whence $|u_+| + |u_-| \geq 1/|z| - 2 > 2$ and it follows that at least one of the roots satisfies $|u| > 1$. By $|u_+| \times |u_-| = 1$, the other root must satisfy

$|u| < 1$. Observe that $|u_+| \to \infty$ as $z \to 0$, thus, by continuity of both roots, we deduce that $|u_+| > 1$ and $|u_-| < 1$ for every $|z| < 1/4$.

Write

$$\frac{1}{u-(1+u)^2 z} = \frac{A}{u-u_-} + \frac{B}{u-u_+}\,,$$

where $A = A(z)$ and $B = B(z)$ actually depend on z.

By Cauchy's Formula, since only u_- lies within the circle $|u| = 1$, we deduce

$$\oint_{|u|=1} \frac{du}{u-(1+u)^2 z} = 2\pi i A\,, \qquad A = \lim_{u\to u_-} \frac{u-u_-}{u-(1+u)^2 z} = \frac{1}{(u-(1+u)^2 z)'|_{u=u_-}}\,.$$

And we can verify that $A = \frac{1}{\sqrt{1-4z}}$. Thus the formula follows.

(c) This part follows by noticing that, by integration of the series

$$\sum_{n=0}^{\infty} \binom{2n}{n} \tfrac{1}{n+1} z^n = \tfrac{1}{z} \int_0^z \left(\sum_{n=0}^{\infty} \binom{2n}{n} u^n \right) du\,,$$

which is valid for $|z| < 1/4$, within the radius of convergence.

Since

$$\int_0^z \left(\sum_{n=0}^{\infty} \binom{2n}{n} u^n \right) du = \int_0^z \frac{du}{\sqrt{1-4u}} = \frac{1-\sqrt{1-4z}}{2}\,,$$

the result follows.

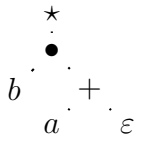
2. **Asymptotics number of regular expressions [Transfer Theorem]***

Consider the combinatorial class $\mathcal{R}$ of regular expressions as unary-binary trees:

(a) leaves are decorated by the letters "a", "b" or the empty word "ε".

(b) internal binary nodes are decorated by the operators union "$+$" or concatenation "$\bullet$",

(c) internal unary-nodes are decorated by the Kleene-star "$\star$".



$(b \cdot (a+\varepsilon))^\star$

Determine the asymptotics for the number of regular expressions with n symbols (nodes).

Responses. The class is specified as

$$\mathcal{R} = a + b + \varepsilon + (\star \times \mathcal{R}) + (''+'' \times \mathcal{R} \times \mathcal{R}) + (''\bullet'' \times \mathcal{R} \times \mathcal{R}).$$

This translates into

$$R(z) = 3z + zR(z) + z(R(z))^2 + z(R(z))^2 = 3z + zR(z) + 2z(R(z))^2\,,$$

which we solve to

$$R(z) = \frac{1 - z - \sqrt{1 - 2z - 23z^2}}{4z}.$$

We note that the $+$ sign in the solution of the quadratic equation is not valid because then $R(z)$ would not be defined a $z = 0$.

The dominant singularity is the point closest to 0 where the argument of the $\sqrt{\cdot}$ is 0. In this case, solving the quadratic equation under the square-root we have two roots:

$$z_0 = (2\sqrt{6} - 1)/23, z_1 = -(2\sqrt{6} + 1)/23.$$

The radius of convergence is $\rho = (2\sqrt{6} - 1)/23$, the singularity closest to the origin.

We factor inside the square root[1]

$$\sqrt{1 - 2z - 23z^2} = \sqrt{(z_0 - z)} \times \sqrt{(z - z_1)} \sim \sqrt{(z_0 - z)}\sqrt{(z_0 - z_1)} \text{ as } z \to z_0$$

because the part $z \mapsto \sqrt{(z - z_1)}$ is analytic on a larger disc, actually on $\mathbb{C} \setminus (-\infty, z_1]$.

It follows that

$$R(z) = \frac{1 - z - \sqrt{1 - 2z - 23z^2}}{4z} \sim \frac{1 - z_0 - \sqrt{z_0(z_0 - z_1)} \times \sqrt{1 - z/z_0}}{4z_0}$$

as $z \to z_0$. From the Transfer Theorem we deduce

$$[z^n]R(z) \sim \frac{-\sqrt{z_0(z_0 - z_1)}}{4z_0}(z_0)^{-n}\frac{n^{-3/2}}{\Gamma(-1/2)} = \frac{\sqrt{z_0(z_0 - z_1)}}{8z_0\sqrt{\pi}}n^{-3/2}(z_0)^{-n},$$

where we remark that[2] $\Gamma(-1/2) = -2\sqrt{\pi}$.

3. **Daffodil Lemma [Localization of singularities]**

   (a) Prove the following classical lemma:

   Daffodil Lemma. Let $f(z)$ be analytic on $|z| < \rho$, with non-negative coefficients at $z = 0$. If the expansion does not reduce to a single monomial and, for some $w \neq 0$, $|w| < \rho$, we have $f(|w|) = |f(w)|$, then

   (a.1) we must have $w = Re^{i\theta}$ with $\theta/(2\pi) = \frac{r}{p}$ a rational, $\gcd(r, p) = 1$, $p > 0$,

   (a.2) the non-zero coefficients $a_n = [z^n]f(z)$ occur at positions n all belonging to $a + p\mathbb{Z}_{\geq 0}$ for some fixed a.

   (b) [The statement has been corrected] Consider the class $\mathcal{C}$ of binary words without K consecutive ones, with $K \in \mathbb{Z}_{\geq 2}$ fixed.

   (b.1) Prove that $C(z) = \frac{1+z+\ldots+z^{K-1}}{1-z\times(1+z+\ldots+z^{K-1})} = \frac{1-z^K}{1-2z+z^{K+1}}$.

   (b.2) Give a polynomial equation characterizing the radius of convergence $z = \rho_C$. Prove that ρ_C is a simple pole of $C(z)$.

   (b.3) Prove that there are no other singularities on $|z| = \rho_C$.

   (b.4) Conclude that $[z^n]C(z) \sim c \cdot \rho_C^{-n}$ and give an expression for c in terms of ρ_C and K.

   Note. The Fibonacci numbers correspond to the case $K = 2$, then $[z^n]C(z) = f_{n+1}$ and $\rho_C = 1/\phi$, $\phi = (1 + \sqrt{5})/2$.

[1] Notice that $\sqrt{A(z)B(z)} = \sqrt{A(z)} \times \sqrt{B(z)}$ is true by analytic continuation whenever we can define the arguments to be non-negative.

[2] This follows from the well-known $\Gamma(1/2) = \sqrt{\pi}$ and $z\Gamma(z) = \Gamma(z + 1)$, setting $z = -1/2$.

Responses.

(a.1) Notice that $|f(w)| \leq f(|w|)$ holds always, due to the triangle inequality. For there to be equality, all terms of the sum in $f(w)$ must "point" in the same direction seen as vectors in the complex plane.

Write $f(z) = \sum a_n z^n$. By hypothesis the coefficients $a_n \geq 0$, and there are at least two $a_{n_1} > 0$ and $a_{n_2} > 0$ with $n_1 < n_2$. By the previous argument we must have that $\arg(a_{n_1} w^{n_1}) = \arg(a_{n_2} w^{n_2})$, the angle is the same. Since the coefficients are positive, this is equivalent to $\arg(w^{n_1}) = \arg(w^{n_2})$. Thus, if $w = Re^{i\theta}$, then $\theta n_1 \equiv \theta n_2 (\mathrm{mod} 2\pi)$. Thus $\theta(n_2 - n_1) = 2\pi m$ for some $m \in \mathbb{Z}$ and we deduce $\theta = 2\pi \frac{m}{n_2 - n_1}$.

(a.2) Here $p > 0$ is the denominator of the reduced fraction $\theta/(2\pi) = r/p$. Let $k = a$ be the smallest index[3] such that $a_k > 0$. Any index n with $a_n > 0$ must satisfy, due to part (a) that $\theta(n - a) \in 2\pi\mathbb{Z}$. Using the equation for θ, we have $\frac{r}{p}(n - a) \in \mathbb{Z}$, but then p must divide $n - a$, because $\gcd(r, p) = 1$. Thus $n \in a + p\mathbb{Z}_{\geq 0}$.

(b.1) The class can be specified as $\mathcal{C} = \mathtt{Seq}(\mathtt{Seq}_{<K}(\mathbf{1}) \cdot \mathbf{0}) \cdot \mathtt{Seq}_{<K}(\mathbf{1})$, where the subscript of the sequences denote a restriction to the involved lengths. This is explained as follows: whenever there is a 0, it must be preceded by less that K 1s, and there might be a tail of at most $K - 1$ $1s$.
Thus it follows that the ordinary generating function is

$$C(z) = \frac{1 + z + \ldots + z^{K-1}}{1 - z \times (1 + z + \ldots + z^{K-1})}\,.$$

Using $1 + z + \ldots + z^{K-1} = \frac{1 - z^K}{1 - z}$ we find an alternative expression

$$C(z) = \frac{1 - z^K}{1 - 2z + z^{K+1}}\,.$$

Note that in this latter expression $z = 1$ is a common root of the denominator and the numerator, and it cancels out as it is a simple root of both.

(b.2) Looking back at the equation for $C(z)$ before simplifying $q_K(z) := 1 + z + \ldots + z^{K-1}$, as long as $|zq_K(z)| < 1$ the function $C(z)$ is analytic. By Pringsheim's Theorem, the radius of convergence of a power series with positive coefficients is itself a dominant singularity. Thus it is enough to look at the positive axes to determine ρ_C.
The singularity comes when $\rho_C \times q_K(\rho_C) = 1$ We note that $\rho_C < 1$ as $1 \times q_K(1) = K > 1$. Since $\rho_C \neq 1$, from the second expression for $C(z)$ we conclude that

$$1 - 2\rho_C + \rho_C^{K+1} = 0\,.$$

The polynomial $p(x) = 1 - 2x + x^{K+1}$ has a derivative $p'(x) = -2 + (K + 1)x^K$. If ρ_C were a double root, then $p'(\rho_C) = 0$ and so $\rho_C^K = \frac{2}{K+1}$. Since $0 = 1 - 2\rho_C + \rho_C^{K+1}$, we deduce $0 = 1 - 2\rho_C + \rho_C \frac{2}{K+1} = 1 - 2\rho_C(1 - 1/(K + 1))$, i.e., $\rho_C = (K + 1)/(2K)$.
But $2/(K + 1) = \rho_C^K$ yields $(K + 1)^{K+1} = 2^{K+1} K^K$. Since $\gcd(K, K + 1) = 1$ and $(K + 1)^{K+1}$ divides $2^{K+1} K^K$, $(K + 1)^{K+1}$ must divide 2^{K+1}. However $K + 1 \geq 3$ and so $(K + 1)^{K+1} > 2^{K+1}$, an absurd. Thus ρ_C is a simple root and the pole is simple.

[3] Sorry for the choice of the letter a...

(b.3) We apply the Daffodil Lemma. Suppose $w = \rho_C e^{i\theta}$ were a dominant singularity. Then again we must have $w \times (1 + w + \ldots + w^{K-1}) = 1$. Let $f(z) = z \times (1 + z + \ldots + z^{K-1})$. We have $|f(w)| = 1 = f(\rho_C) = f(|w|)$. Thus we are ready to apply the Lemma: $\theta = 2\pi \cdot r/p$ for some reduced fraction with $\gcd(r, p) = 1$ and $p > 0$. But $(a.2)$ implies that, then, all non-zero coefficients of $f(z)$ belong to $a + p\mathbb{Z}_{\geq 0}$ for some a. Looking at $f(z)$, this is only possible for $p = 1$, and it follows that $\theta \in 2\pi\mathbb{Z}$ and $w = \rho_C$. Thus there are no other dominant singularities.

(b.4) The generating function $C(z)$ is rational and $\rho_C > 0$ is the only dominant singularity. We proved that the pole at $z = \rho_C$ is simple, and then $C(z) \sim \frac{c}{1 - z/\rho_C}$ for some $c \neq 0$, as $z \to \rho_C$. By the principle of asymptotics of rational functions (see Section 3.2 of the notes), or the Transfer Theorm (a bit overkill, but true), $[z^n]C(z) \sim c \cdot \rho_C^{-c}$.

The constant c must equal

$$\lim_{z \to \rho_C} (1 - z/\rho_C) C(z) = -\tfrac{1}{\rho_C} \lim_{z \to \rho_C} (z - \rho_C) \frac{1 - z^K}{p(z)} = -\frac{1 - \rho_C^K}{\rho_C} \frac{1}{p'(\rho_C)} = \frac{1 - \rho_C^K}{\rho_C} \frac{1}{2 - (K+1)\rho_C^K}.$$

Extra. Since $c > 0$ by combinatorial considerations, and $\rho_C^K < 1$, we deduce that $2 > (K+1)\rho_C^K$, i.e., $\sqrt[K]{2/(K+1)} > \rho_C$.