Analytic Combinatorics of Unlabeled Objects

Set of exercises III: responses

1. **Multivariate Generating Functions.**

   A) Integer compositions* We recall that integer compositions are specified by

   $$\mathcal{C} = \mathtt{Seq}(\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots\}) \simeq \mathtt{Seq}(\mathtt{Seq}_{\geq 1}(\mathcal{Z}))\,.$$

   During class, we used OGFs to prove that the average number of terms in a composition of n is $\sim n/2$.

   (a) What if we restricted the compositions to have an even number of terms? Find the total number of compositions of n into an even number of terms.

   (b) Let us consider now the number of even terms in a general composition of n. Show that the average number of even terms is $\sim n/6$.

   (c) Now we want to fix the number m of even terms. Find the generating function.

   Responses.

   (a) This can be done by simply considering a constrained "Seq" constructor in which we just consider the even powers. We obtain

   $$\mathcal{C}_2 = \mathtt{Seq}_{\mathtt{even}}(\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots\}) \Longrightarrow C_2(z) = \frac{1}{1 - \left(\frac{z}{1-z}\right)^2} = \frac{(1-z)^2}{1-2z} = 1 + \frac{z^2}{1-2z}\,.$$

   Another possible way to see this is to mark the number of terms and extract the even terms, or even considering that $\mathcal{C}_2 = \mathtt{Seq}(\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots\} \times \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots\})$, a sequence of pairs.
   For the total number of partitions into even parts we obtain, for $n \geq 2$

   $$[z^n]C_2(z) = [z^n]\frac{z^2}{1-2z} = [z^{n-2}]\frac{1}{1-2z} = 2^{n-2}\,.$$

   We note that $[z^0]C_2(z) = 1$ and $[z^1]C_2(z) = 0$.

   (b) We mark the even terms

   $$\mathcal{C} = \mathtt{Seq}\left(\mathcal{U} \times \{\mathbf{2}, \mathbf{4}, \mathbf{6}, \ldots\} + \{\mathbf{1}, \mathbf{3}, \mathbf{5}, \ldots\}\right) \Longrightarrow C(z,u) = \frac{1}{1 - u\frac{z^2}{1-z^2} - \frac{z}{1-z^2}} = \frac{1-z^2}{1-z-(1+u)\,z^2}\,.$$

   Differentiating in u and then setting $u = 1$ we obtain

   $$\partial_u C(z,1) = z^2 \frac{1-z}{(1+z)(1-2z)^2}\,.$$

   Apply partial fractions

   $$\frac{1-z}{(1+z)(1-2z)^2} = \frac{1/3}{(1-2z)^2} + \frac{-4/9}{1-2z} + \frac{2/9}{1+z}\,.$$

Extracting coefficients and noting that the dominant terms come from $\frac{1/3}{(1-2z)^2}$,

$$[z^n]\partial_u C(z,1) = [z^{n-2}]\frac{1-z}{(1+z)(1-2z)^2} \sim [z^{n-2}]\frac{1/3}{(1-2z)^2} = \tfrac{1}{3}(n-1)2^{n-2}\,.$$

Since $[z^n]C(z,1) = 2^{n-1}$, dividing we obtain $\frac{[z^n]\partial_u C(z,1)}{[z^n]C(z,1)} \sim \frac{1}{6}n$.

(c) The generating function we look for is $F_m(z) = [u^m]C(z,u)$. Thus, expanding the binomials and summing the contributions

$$F_m(z) = [u^m]\sum_{k=0}^{\infty}\left(u\tfrac{z^2}{1-z^2} + \tfrac{z}{1-z^2}\right)^k = \sum_{k=0}^{\infty}\binom{k}{m}\tfrac{z^{2m}}{(1-z^2)^m}\left(\tfrac{z}{1-z^2}\right)^{k-m}\,.$$

We simplify the latter expression by using $\sum\binom{k}{m}u^k = \frac{u^m}{(1-u)^{m+1}}$,

$$F_m(z) = z^m\sum_{k=0}^{\infty}\binom{k}{m}\left(\tfrac{z}{1-z^2}\right)^k = z^m\frac{\left(\frac{z}{1-z^2}\right)^m}{\left(1-\left(\frac{z}{1-z^2}\right)\right)^{m+1}} = \frac{z^{2m}}{(1-z-z^2)^{m+1}}\,.$$

B) Binary trees

We consider the binary trees

$$\mathcal{B} = \mathcal{E} + \mathcal{Z}\times\mathcal{B}\times\mathcal{B}\,,$$

where the node $\mathcal{E}$ is the empty-node, while $\mathcal{Z}$ is an atom of weight 1.

(a) Give a specification for the binary trees $\mathcal{B}^\circ$ marking the length of the left-most branch.

(b) Solve the equation for $B(z,u)$ and show $\partial_u B(z,1) = z(B(z))^3$.

(c) Let $A(z) = zB(z)$. Apply the Lagrange Inversion Formula to compute $[z^n](A(z))^3$.

(d) Deduce that the average length of the left-most branch tends to 3 as $n\to\infty$.

(e) Generalize. Find the exact number of binary trees of size n with left-most branch of length m.

Responses.

(a) We consider $\mathcal{B}$ as given, and we have $\mathcal{B}^\circ = \mathcal{E} + \mathcal{Z}\times\mathcal{U}\times\mathcal{B}^\circ\times\mathcal{B}$.
Another specification is the following:

$$\mathcal{B}^\circ = \mathsf{Seq}(\mathcal{Z}\times\mathcal{U}\times\mathcal{B})\,,$$

by seeing the left-most branch as a list with binary trees pending on the right of each node.

(b) We have $B(z,u) = \frac{1}{1-zuB(z)}$. Differentiating we find $\partial_z B(z,1) = \frac{zB(z)}{(1-zB(z))^2}$.
Recall that $B(z) = 1 + z(B(z))^2$, thus we derive $B(z) = \frac{1}{1-zB(z)}$. We deduce from this that $\partial_z B(z,1) = z(B(z))^3$.

(c) We have $A(z) = z\phi(A(z))$ with $\phi(u) = \frac{1}{1-u}$. Indeed $B(z) = \frac{1}{1-zB(z)}$. We deduce, applying the LIF with $f(u) = u^3$ and $\phi(u) = \frac{1}{1-u}$,

$$[z^n]A(z)^3 = \tfrac{[u^{n-1}]}{n}\{3u^2(\phi(u))^n\} = \tfrac{3}{n}[u^{n-3}]\{\tfrac{1}{(1-u)^n}\} = \tfrac{3}{n}\binom{n-1+n-3}{n-3}.$$

Thus $[z^n]A(z)^3 = \frac{3}{n}\binom{2n-4}{n-3}$.

(d) We have $[z^n]\partial_u B(z,1) = [z^n]zB(z)^3 = [z^{n+2}]A(z)^3 = \frac{3}{n+2}\binom{2n}{n-1}$ while $[z^n]B(z) = \binom{2n}{n}\frac{1}{n+1}$. Dividing we obtain the average length:

$$3\frac{n+1}{n+2}\frac{\binom{2n}{n-1}}{\binom{2n}{n}} = 3\frac{n+1}{n+2} \times \frac{n}{n+1} = 3\frac{n}{n+2} \to 3\,.$$

(e) We extract $[u^m]B(z,u) = [u^m]\frac{1}{1-zuB(z)} = (zB(z))^m$, which is the generating function of the trees having left-most branch of length m. We apply the same argument to $A(z)$. The total number of trees of size n with left-most branch of length m is:

$$[z^n]A(z)^m = \frac{m}{n}[u^{n-m}]\{\tfrac{1}{(1-u)^n}\} = \frac{m}{n}\binom{n-1+n-m}{n-m}$$

2. The Kraft-McMillan inequality

In this exercise we prove a key inequality in Coding and Information Theory by using the radius of convergence.

Let $\mathcal{A} = \{a_1, \ldots, a_k\}$ and $\mathcal{B} = \{\mathbf{0}, \mathbf{1}\}$. A **binary code** $c\colon \mathcal{A}^\star \to \mathcal{B}^\star$ is a morphism, i.e., $c(xy) = c(x)c(y)$. The code c is *uniquely decodable* iff it is injective.

(a) Let[1] $A(z) = \sum_{a\in\mathcal{A}} z^{|c(a)|}$. Show $C(z) = \sum_{v\in c(\mathcal{A}^\star)} z^{|v|} = \frac{1}{1-A(z)}$.

(b) [**Kraft-McMillan inequality**] Prove that if c is uniquely decodable, then

$$\sum_{a\in\mathcal{A}} 2^{-|c(a)|} \le 1\,.$$

Hint. Reason by contradiction. Note that power-series are continuous within their radius of convergence.

(c) [**Shannon bound**] (Optional) Consider a distribution $\mathbf{p} = (p_1, \ldots, p_k)$ over $(a_1, \ldots, a_k)$. Prove that, for a random symbol X,

$$H(\mathbf{p}) = \sum_{j=1}^{k} p_j \log_2(1/p_j) \le \sum_{j=1}^{k} p_j |c(a_j)| = \mathbb{E}[c(X)]\,.$$

Hint. Prove that $\log x \le x - 1$ for all $x \in \mathbb{R}_{>0}$. Use the Kraft-McMillan inequality.

[1] Here $|w|$ is the length of the word, i.e., the number of symbols.

Response.

(a) Since c is a morphism, any element $v \in c(\mathcal{A}^\star)$ can be seen as a *sequence* of elements from $c(\mathcal{A})$, whose OGF is $A(z)$, thus $C(z) = \frac{1}{1-A(z)}$.

(b) Suppose otherwise. Then $A(1/2) = \sum_{a\in\mathcal{A}} 2^{-|c(a)|} > 1$.

– By continuity of $A(z)$ within its radius of convergence, which is infinite since $A(z)$ is actually a polynomial because $\mathcal{A}$ is finite, it follows that there must be $c \in (0, 1/2)$ such that $A(c) = 1$. We suppose that c is the smallest such $c > 0$. Then $A(t) < 1$ for $0 \le t < c$ and $C(z)$ converges there. Observe that increases $C(t) \to \infty$ as $t \to c^-$.

– Being c uniquely decodable, it follows that there must be at most 2^n words of length n in $c(\mathcal{A}^\star)$ for each $n \ge 0$. Hence, for $t \in (0, 1/2)$, we deduce $C(t) \le \sum_{n=0}^{\infty} 2^n t^n = \frac{1}{1-2t}$.This means that $C(t) \le \frac{1}{1-2c} < \infty$ for $0 \le t < c$, a contradiction.

(c) We first prove the inequality $\log x \le x - 1$ for all $x > 0$. Considering $f(x) = x - 1 - \log x$, we have $f'(x) = 1 - \frac{1}{x}$. Looking at the sign of $f'(x)$, we deduce that $f(x)$ decreases on $(0, 1]$ and increases in $[1, \infty)$. Hence the minimum of $f(x)$ is found at $x = 1$, $f(x) \ge f(1) = 0$ and the auxiliary inequality is proved.

The inequality we want to prove, $H(\mathbf{p}) \le \mathbb{E}[c(X)]$, is equivalent to

$$\sum_j p_j \log_2(2^{-|c(a_j)|}/p_j) \le 0\,.$$

By using the inequality $\log(x) \le x-1$ we deduce $\log_2(2^{-|c(a_j)|}/p_j) \le \frac{1}{\log 2}\left(2^{-|c(a_j)|}/p_j - 1\right)$, thus

$$\sum_j p_j \log_2(2^{-|c(a_j)|}/p_j) \le \frac{1}{\log 2}\sum_j p_j \left(2^{-|c(a_j)|}/p_j - 1\right) = \frac{1}{\log 2}\left(\sum_j 2^{-|c(a_j)|} - \sum_j p_j\right) = 0\,,$$

where the last equality follows from the Kraft-inequality $\sum_j 2^{-|c(a_j)|} \le 1$ and the equality $\sum_j p_j = 1$. Thus the result is proved.

3. **Meromorphic asymptotics*.** For each of the following functions $f(z)$, find the radius of convergence and an equivalent (asymptotic) for the coefficients of the power-series expansion $f(z) = \sum a_n z^n$ at $z = 0$.

(a) $f(z) = \frac{1}{(1-2z)^2}\log\left(\frac{1}{1-z}\right)$.

(b) $f(z) = \frac{e^{2z}-1}{(1-z)(1-2z)^3}$.

(c) $f(z) = \frac{1}{2-e^z}$.

Response.

(a) The singularity closest to the origin is $z = 1/2$, as $\log\left(\frac{1}{1-z}\right)$ is analytic on $|z| < 1$. Thus the radius of convergence is $\rho = 1/2$. For the asymptotics, expanding $\log(\frac{1}{1-z})$ around $z = 1/2$, we obtain $f(z) = \frac{1}{(1-2z)^2}\log\left(\frac{1}{1-1/2}\right) + \frac{1}{(1-2z)^2}\left(\frac{1}{1-1/2}\right)(z - 1/2) + \texttt{analytic}$.

That is

$$g(z) = f(z) - \left(\tfrac{1}{(1-2z)^2}\log 2 - \tfrac{1}{1-2z}\right)$$

is analytic at $z = 1/2$, and hence analytic on $|z| < 1$. It follows that, for any fixed $\epsilon > 0$, we have $[z^n]g(z) = O((1+\epsilon)^n)$. Thus

$$[z^n]f(z) = [z^n]\tfrac{1}{(1-2z)^2}\log 2 + [z^n]\tfrac{1}{1-2z} + O((1+\epsilon)^n) = (\log 2)\cdot(n+1)\cdot 2^n - 2^n + O((1+\epsilon)^n)\,.$$

We deduce $[z^n]f(z) \sim (\log 2)n2^n$.

(b) The singularity closest to the origin is $z = 1/2$. Thus the radius of convergence is $\rho = 1/2$. In this case, expanding $\frac{e^{2z}-1}{1-z}$ around $z = 1/2$ we obtain

$$f(z) = \frac{2(e-1)}{(1-2z)^3} + \frac{c_1}{(1-2z)^2} + \frac{c_2}{(1-2z)^1} + \text{analytic at } z = 1/2\,.$$

The latter analytic function, being analytic up to $z = 1$, it will not contribute to the dominant asymptotics. Since $[z^n](1-z/\rho)^{-m} \sim \frac{n^{m-1}}{(m-1)!}\rho^{-n}$ for any positive integer $m \geq 1$,

$$[z^n]f(z) \sim 2(e-1)\frac{n^2}{2}2^n = (e-1)n^2 2^n\,.$$

(c) The singularities satisfy $e^z = 2$, i.e., these are $z = \log 2 + 2\pi i k$ with $k \in \mathbb{Z}$. The closest to the origin is $\log 2$. Thus the radius of convergence is $\rho = \log 2$.
Expand the power series of the denominator around $z = \log 2$,

$$\frac{1}{2-e^z} = \frac{1}{(2-e^z)'_{z=\log 2}(z-\log 2) + \sum_{k\geq 2} c_k(z-\log 2)^k} = \frac{1/2}{\log 2 - z} \times \frac{1}{1+\sum_{k\geq 1}\hat{c}_k(z-\log 2)^k}\,.$$

The multiplicative inverse $(1+\sum_{k\geq 1}\hat{c}_k(z-\log 2)^k)^{-1}$ is well-defined, since the denominator is not 0 at $z = \log 2$, and $(1+\sum_{k\geq 1}\hat{c}_k(z-\log 2)^k)^{-1} = 1 + \sum_{k\geq 1} d_k(z-\log 2)^k$ for certain coefficients. Remark, moreover, that this expansion is entire (radius of convergence ∞) as this is the case for $2 - e^z$.
Expanding

$$f(z) = \frac{1/2}{\log 2 - z} + \texttt{analytic at } z = \log 2\,.$$

Since there are no more singularities on $|z| = \log 2$ and the function is meromorphic we deduce that

$$[z^n]f(z) \sim [z^n]\frac{1/2}{\log 2 - z} = [z^n]\frac{1/(2\log 2)}{1 - z/\log 2} = \tfrac{1}{2}(\log 2)^{-(n+1)}\,.$$