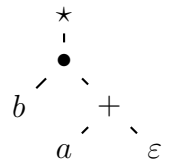
Analytic Combinatorics of Unlabeled Objects

Set of exercises II: responses

1. **Warm up: regular expressions***

   Consider the combinatorial class $\mathcal{R}$ of regular expressions as unary-binary trees:

   - leaves are decorated by the letters "a", "b" or the empty word "ε".
   - internal binary nodes are decorated by the operators union "$+$" or concatenation "$\bullet$",
   - internal unary-nodes are decorated by the Kleene-star "$\star$".

   $$(b \cdot (a + \varepsilon))^\star$$

   Give a combinatorial specification. Determine the ordinary generating function supposing that the size is the total number of nodes.

   Response. The specification is

   $$\mathcal{R} = \{a, b, \varepsilon\} + \{\star\} \times \mathcal{R} + \{+\} \times \mathcal{R} \times \mathcal{R} + \{\bullet\} \times \mathcal{R} \times \mathcal{R}\,.$$

   This translates into an equation for the OGF $R(z)$,

   $$R(z) = 3z + zR(z) + 2z(R(z))^2\,,$$

   since here every node (even the leaves) have size 1.

   Solving the equation we obtain:

   $$R(z) = \frac{1 - z - \sqrt{1 - 2z - 23z^2}}{4z}\,,$$

   noting that the other root does not make sense at $z = 0$.

2. **Multivariate Generating Functions**

   Postponed to TD3.

3. **Rational asymptotics**

   Find the generating function*

   (a) Find the ordinary generating function of the class of words on $\{a, b\}$ such that, between any two b's there is at least one a.

   (b) Consider the class of words on $\{a, b, c\}$ such that

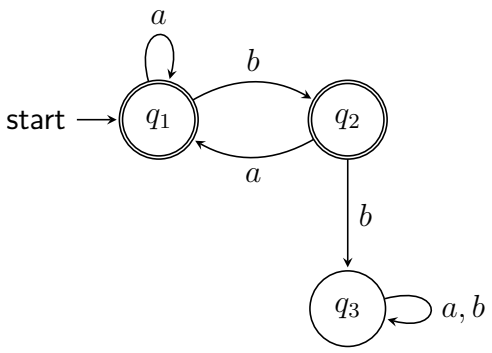
   $$\#a + \#b + 2\#c \equiv 0 \pmod 3\,.$$

   Write a formula for the OGF in terms of a matrix.

Response. (a) This regular languages can be specified unambiguously as $a^* + a^*(ba^+)^*ba^*$, in regular expressions, or $\mathsf{Seq}(a) + \mathsf{Seq}(a) \times \mathsf{Seq}(b \times \mathsf{Seq}_{\geq 1}(a)) \times \{b\} \times \mathsf{Seq}(a)$ in the notation from the book *Analytic Combinatorics*. This translates into

$$\frac{1}{1-z} + \frac{1}{1-z}\frac{1}{1-\frac{z^2}{1-z}}z\frac{1}{1-z} = \frac{1}{1-z} + \frac{1}{1-z}\frac{z}{1-z-z^2} = \frac{1}{1-z}\frac{1-z^2}{1-z-z^2} = \frac{1+z}{1-z-z^2}.$$

Another way to do this: we consider the DFA



The associated transition matrix is $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ where entry (i, j) denotes the number of transitions going from q_i to q_j. We recall that $[A^k]_{i,j}$ is then the number of paths (with possible repetitions of nodes) of length k from q_i to q_j.

Thus, the OGF we are looking for is

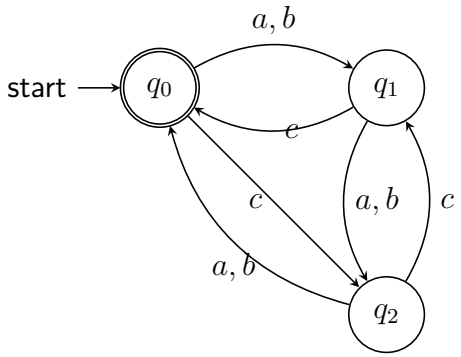
$$[\sum_k A^k z^k]_{1,1} + [\sum_k A^k z^k]_{1,2} = (1\,0\,0) \times \sum_k A^k z^k \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

as q_1 is the initial state and q_1 and q_2 the accepting states. Since $\sum_k A^k z^k = (\mathtt{Id} - zA)^{-1}$, with *lots of pain* (or rather, a computer) we deduce

$$(\mathtt{Id} - zA)^{-1} = \begin{pmatrix} \frac{1}{1-z-z^2} & \frac{z}{1-z-z^2} & \frac{z^2}{2z^3+z^2-3z+1} \\ \frac{z}{1-z-z^2} & \frac{1-z}{1-z-z^2} & \frac{z-z^2}{2z^3+z^2-3z+1} \\ 0 & 0 & \frac{1}{1-2z} \end{pmatrix},$$

summing the entries $(1, 1)$ and $(1, 2)$ we find the same OGF.

(b) The DFA is the following, corresponding to the sum modulo 3:



Thus we obtain the matrix

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix},$$

and the inverse is

$$(\mathtt{Id} - zA)^{-1} = \begin{pmatrix} \frac{1-2z^2}{1-6z^2-9z^3} & \frac{z^2+2z}{1-6z^2-9z^3} & \frac{4z^2+z}{1-6z^2-9z^3} \\ \frac{4z^2+z}{1-6z^2-9z^3} & \frac{1-2z^2}{1-6z^2-9z^3} & \frac{z^2+2z}{1-6z^2-9z^3} \\ \frac{z^2+2z}{1-6z^2-9z^3} & \frac{4z^2+z}{1-6z^2-9z^3} & \frac{1-2z^2}{1-6z^2-9z^3} \end{pmatrix},$$

therefore the response is $\frac{1-2z^2}{1-6z^2-9z^3}$.

Schur's Theorem Prove the following theorem:

Schur's Theorem. If c_n represents the number of representations of n as a non-negative integer combination of $a_1, \ldots, a_M$, these being a set of positive integers with $\gcd(a_1, \ldots, a_M) = 1$, then

$$c_n \sim \frac{n^{M-1}}{(M-1)!\, a_1 \ldots a_M}.$$

Response. The representations correspond to multisets of $a_1, a_2, \ldots, a_M$ with sizes $|a_i| = a_i$. Thus the generating function is

$$C(z) = \sum c_n z^n = \frac{1}{1-z^{a_1}} \cdots \frac{1}{1-z^{a_M}}.$$

We perform partial fractions. Note that all of the roots ζ of the denominator $Q(z) = \prod_{j=1}^{M}(1 - z^{a_j})$ are roots of unity with $\zeta^{a_j} = 1$ for some a_j.

The roots of each $(1 - z^{a_j})$ are simple (it suffices to take the derivative), hence, the multiplicity of a root ζ of $Q(z)$ is the number of terms j such that $\zeta^{a_j} = 1$. Of course, $\zeta = 1$ has multiplicity M. We prove that all other roots have multiplicity strictly less than M. Indeed, write $\zeta = \exp(2\pi i \frac{p}{q})$. Observe that if $\zeta^{a_j} = 1$, then $q | a_j$. If this were the case for every j we would have that q is a common divisors of all a_j and hence $q = 1$.

Thus we may write

$$C(z) = \frac{A}{(1-z)^M} + \sum_{\zeta : Q(\zeta)=0} \sum_{k=1}^{M-1} \frac{\alpha_{\zeta,k}}{(1-z/\zeta)^k},$$

for some coefficients A and $\alpha_{\zeta,k}$. We compute A by multiplying both sides by $(1-z)^M$ and letting $z \to 1$, to obtain $A = \frac{1}{a_1 a_2 \ldots a_M}$. T

hen we remark that

$$[z^n] \frac{\alpha_{\zeta,k}}{(1-z/\zeta)^k} = \alpha_{\zeta,k} \binom{k+n-1}{n-1} \zeta^{-n} \sim \alpha_{\zeta,k} \frac{n^{k-1}}{(k-1)!} \zeta^{-n}.$$

Being $|\zeta| = 1$ for all roots of $Q(z)$ and $k < M$, it follows that these coefficients are all negligible with respect to $[z^n] \frac{A}{(1-z)^M} \sim A \frac{n^{M-1}}{(M-1)!}$.

Thus

$$[z^n]C(z) \sim A\frac{n^{M-1}}{(M-1)!} = \frac{n^{M-1}}{(M-1)!\,a_1 \dots a_M}\,.$$

Note, in particular, that all coefficients are positive for large enough n.