Analytic Combinatorics of Unlabeled Objects

Set of exercises I: responses

1. Using generating functions prove the following:

   (a) For every fixed $p \in \mathbb{Z}_{\geq 0}$, $[z^n]\frac{z^p}{(1-z)^{p+1}} = \binom{n}{p}$ for all $n \geq 0$.

   (b) $\sum_{j=0}^{n} \binom{j}{p} = \binom{n+1}{p+1}$. [Hockey-stick identity]

   (c) $\sum_j \binom{n}{j}\binom{m}{k-j} = \binom{n+m}{k}$. [Vandermonde's identity]

Responses.

(a) There are many possible proofs. Observe that, differentiating p times $\frac{1}{1-z} = 1+z+z^2+\ldots$ we have

$$\frac{p!}{(1-z)^{p+1}} = \sum (n+p)(n+p-1)\ldots(n+1)z^n\,.$$

Therefore

$$\frac{z^p}{(1-z)^{p+1}} = \sum \frac{(n+p)(n+p-1)\ldots(n+1)}{p!} z^{n+p} = \sum \frac{n(n-1)\ldots(n-p+1)}{p!} z^n\,.$$

Since $\frac{n(n-1)\ldots(n-p+1)}{p!} = \binom{n}{p}$, we are done.

(b) Recalling that if $A(z) = \sum a_n z^n$ we have $\frac{1}{1-z}A(z) = \sum_n (\sum_{j\leq n} a_j) z^n$, the left-hand side is the application to $A(z) = \frac{z^p}{(1-z)^{p+1}}$, the OGF of $(\binom{n}{p})_n$.

Thus we obtain $[z^n]\frac{1}{1-z}\frac{z^p}{(1-z)^{p+1}} = [z^n]\frac{z^p}{(1-z)^{p+2}} = [z^{n+1}]\frac{z^{p+1}}{(1-z)^{p+2}} = \binom{n+1}{p+1}$.

(c) Multiply through by z^k and sum over k. We use the usual convention (in order to avoid the limits of summation) that we sum over $k \in \mathbb{Z}$, $j \in \mathbb{Z}$. Note that $\binom{a}{b} = 0$ for $b > a$ or $b < 0$. Thus the expressions make sense.

We obtain:

$$\begin{aligned}
\sum_k \left(\sum_j \binom{n}{j}\binom{m}{k-j} \right) z^k &= \sum_j \binom{n}{j} \sum_k \binom{m}{k-j} z^k \\
&= \sum_j \binom{n}{j} z^j \sum_k \binom{m}{k-j} z^{k-j} \\
&= \sum_j \binom{n}{j} z^j (1+z)^m \\
&= (1+z)^{n+m} \\
&= \sum_k \binom{n+m}{k} z^k\,.
\end{aligned}$$

2. In this exercise we prove the Generalized Binomial Theorem: for every $\alpha \in \mathbb{R}$

$$[z^n](1+z)^\alpha = \binom{\alpha}{n} := \frac{\alpha \dots (\alpha - n + 1)}{n!}, \quad \text{for all } n \geq 0.$$

Fix $\alpha \in \mathbb{R}$ and define $f(z) = (1+z)^\alpha$.

(a) Show that $(1+z)f'(z) = \alpha f(z)$ is satisfied.

(b) Deduce a recurrence for $f_n := [z^n]f(z)$ and prove the formula for f_n by induction, thereby proving the Generalized Binomial Theorem.

(c) Use the Generalized Binomial Theorem to prove that the coefficients of

$$B(z) = \frac{1 - \sqrt{1-4z}}{2z}$$

are the Catalan numbers $[z^n]B(z) = \binom{2n}{n}\frac{1}{n+1}$.

Responses. Part (a) is a direct verification $f'(z) = \alpha(1+z)^{\alpha-1}$ and so $(1+z)f'(z) = \alpha(1+z)^\alpha = \alpha f(z)$.

Reading the coefficients we have $[z^n](1+z)f'(z) = (n+1)f_{n+1} + nf_n$ for $n \geq 0$. Thus we deduce $(n+1)f_{n+1} + nf_n = \alpha f_n$ and $f_{n+1} = \frac{\alpha - n}{n+1} f_n$.

Clearly $f_0 = 1 = \binom{\alpha}{0}$ by definition. Assume the formula to hold for f_n, then

$$f_{n+1} = \frac{\alpha - n}{n+1} f_n = \frac{\alpha - n}{n+1}\binom{\alpha}{n} = \frac{\alpha \dots (\alpha - n)}{(n+1)!},$$

which is $\binom{\alpha}{n+1}$ by definition. Thus the Generalized Binomial Theorem follows by induction and part (b) is done.

For part (c), the formula clearly holds for $n = 0$, and we have $[z^n]B(z) = -\frac{1}{2}[z^{n+1}]\sqrt{1-4z}$ for $n \geq 1$.

Now, by the Generalized Binomial Theorem,

$$\begin{aligned}
[z^{n+1}]\sqrt{1-4z} &= (-4)^{n+1}\binom{1/2}{n+1} \\
&= (-4)^{n+1}\frac{\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-n)}{(n+1)!} \\
&= (-4)^{n+1}(-1)^n 2^{-n-1}\frac{(2n-1)(2n-3)\dots 1}{(n+1)!} \\
&= -2^{n+1}\frac{(2n-1)(2n-3)\dots 1}{(n+1)!} \\
&= \frac{2n(2n-2)\dots 2}{2^n n!} \times \left(-2^{n+1}\frac{(2n-1)(2n-3)\dots 1}{(n+1)!}\right) \\
&= -2\frac{(2n)!}{n!n!}\frac{1}{n+1},
\end{aligned}$$

and we are done.

3. The Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ count the number of partitions of a set of n elements into k non-empty subsets. Without loss of generality, we suppose the set of n elements is $[n] = \{1, \ldots, n\}$.

   Prove the following identities

   (a) $\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$ for all[1] $n, k \geq 0$.

   (b) $\sum_{n \geq 0} \left\{ {n \atop k} \right\} z^n = \frac{z^k}{(1-z)(1-2z)\ldots(1-kz)}$.

   Find a formula for $\left\{ {n \atop k} \right\}$ by applying partial fractions.

   Response.

   (a) In a partition of $[n] = \{1, \ldots, n\}$ into k parts, either n is alone making its own part, or not.

   – The total number of partitions in which n is alone is $\left\{ {n-1 \atop k-1} \right\}$, as the rest is a partition of $[n-1]$ into $k-1$ parts.

   – The total number of partitions in which n is not alone is $k \times \left\{ {n-1 \atop k} \right\}$. Indeed, erasing n we have a partition of $[n-1]$ into k parts. Then we must decide to which of these k parts n belongs to, hence the factor k.

   (b) Multiply both sides of the recurrence by k and sum over $k \in \mathbb{Z}$, with the convention $\left\{ {a \atop b} \right\} = 0$ if $a < b$ or $a < 0$. We obtain

$$\begin{aligned} \sum_n \left\{ {n \atop k} \right\} z^n &= \sum_n \left\{ {n-1 \atop k-1} \right\} z^n + k \sum_n \left\{ {n-1 \atop k} \right\} z^n \\ &= z \sum_n \left\{ {n-1 \atop k-1} \right\} z^{n-1} + kz \sum_n \left\{ {n-1 \atop k} \right\} z^{n-1} . \end{aligned}$$

   Letting $A_k(z) = \sum_n \left\{ {n \atop k} \right\} z^k$, we obtain

$$A_k(z) = zA_{k-1}(z) + kzA_k(z) \Longrightarrow A_k(z) = \frac{z}{1-kz} A_{k-1}(z) .$$

   Iterating until $k = 0$, $A_0(z) = 1$ and we have

$$A_k(z) = \frac{z}{1-z} \frac{z}{1-2z} \cdots \frac{z}{1-kz} ,$$

   as desired.

4. In this exercise we give a combinatorial interpretation to

$$\sum_{n \geq 0} \left\{ {n \atop k} \right\} z^n = \frac{z^k}{(1-z)(1-2z)\ldots(1-kz)} .$$

   We define an algorithm. Consider a partition $P = \{S_1, \ldots, S_k\}$:

[1] We define $\left\{ {n \atop k} \right\} = 0$ if every $n < 0$, $k < 0$ or $n < k$.

- We keep an list L of the *known* parts from P. Initially $L = []$.
- We iterate $j = 1, \ldots, n$. For iteration j, let $S_j \in P$ with $j \in S_j$. If S_j appears in L, write its index. If not, append it and write $|V| + 1$.

The *numbers written* belong to $[k]$. They constitute the *backbone*

$$P = \{\{4, 6, 7\}, \{1, 3\}, \{2, 5\}\} \mapsto L(P) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{3} & \mathbf{2} & \mathbf{3} & \mathbf{3}. \end{pmatrix}$$

Prove that this yields an injection into $[k]^n$. Find a combinatorial specification for the valid sequences from $[k]^n$ and deduce the OGF of the partitions into k parts, k fixed.

Response. Essentially, reading the numbers in $L(P)$ gives back the partition. We remark that we have numbered the elements of the partition according to the order given by the minimum element in each part. A valid backbone $L(P)$, then, must have the first appearance of j before the appearance of $j+1$, for each $j \in [k]$. Given any sequence with this properties, it corresponds to a unique partition and vice-versa. Thus we obtain a bijection. Put another way, the numbers in $L(P)$ label the parts of the partition by their order of discovery when reading $j = 1, 2, \ldots, n$.

We note that $L(P)$ can be specified as

$$\{1\} \times \mathtt{Seq}(\{1\}) \times \{2\} \times \mathtt{Seq}(\{1, 2\}) \times \ldots \times \{k\} \times \mathtt{Seq}(\{1, \ldots, k\}),$$

where each number has weight 1.

The corresponding OGF is precisely

$$\frac{z^k}{(1-z)(1-2z)\ldots(1-kz)},$$

as desired.