

Analytic Combinatorics of Unlabeled Objects

Set of exercises 4

October 12, 2026

1. The generating function of the central coefficient

In this exercise we use complex-integration to prove that

$$\sum_{n=0}^{\infty} \binom{2n}{n} z^n = \frac{1}{\sqrt{1-4z}},$$

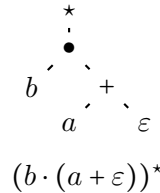
for $|z| < 1/4$.

1. Prove that $F(z) \triangleq \sum_{n=0}^{\infty} \binom{2n}{n} z^n$ converges for $|z| < \frac{1}{4}$.
2. Fix $r > 0$. Remark that $\oint_{|u|=r} \frac{(1+u)^{2n}}{u^{n+1}} du = 2\pi \binom{2n}{n}$. Use this to prove $F(z) = \frac{1}{\sqrt{1-4z}}$.
3. Deduce that $\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{n+1} z^n = \frac{1-\sqrt{1-4z}}{2z}$.

2. Asymptotics number of regular expressions [Transfer Theorem]*

Consider the combinatorial class $\mathcal{R}$ of regular expressions as unary-binary trees:

- leaves are decorated by the letters “a”, “b” or the empty word “ ε ”.
- internal binary nodes are decorated by the operators union “+” or concatenation “ $\bullet$ ”,
- internal unary-nodes are decorated by the Kleene-star “ $\star$ ”.



Determine the asymptotics (an equivalent) for the number of regular expressions with n symbols (nodes).

3. Daffodil Lemma [Localization of singularities]

a) Prove the following classical lemma:*

Lemma 1 (Daffodil Lemma). *Let $f(z)$ be analytic on $|z| < \rho$, with non-negative coefficients at $z = 0$. If the expansion does not reduce to a single monomial and, for some $w \neq 0$, $|w| < \rho$, we have $f(|w|) = |f(w)|$, then*

1. *we must have $w = Re^{i\theta}$ with $\theta/(2\pi) = \frac{r}{p}$ a rational, $\gcd(r, p) = 1$,*

2. the non-zero coefficients $a_n = [z^n]f(z)$ occur at positions n all belonging to $a + p\mathbb{Z}_{\geq 0}$ for some fixed a .

b) Consider the class $\mathcal{C}$ of binary words with at most K consecutive ones, with $K \in \mathbb{Z}_{\geq 2}$ fixed.

1. * Prove that $C(z) = \frac{1}{1-z\frac{1-z^K}{1-z}}$.

2. * Give a polynomial equation characterizing the radius of convergence $z = \rho_C$. Prove that ρ_C is a simple pole of $C(z)$.

3. Prove that there are no other singularities on $|z| = \rho_C$.

4. Conclude that $[z^n]C(z) \sim c \cdot \rho_C^{-n}$ and give an expression for c in terms of ρ_C and K .

Note. The Fibonacci numbers correspond to the case $K = 2$, then $[z^n]C(z) = f_{n+1}$ and $\rho_C = 1/\phi$, $\phi = (1 + \sqrt{5})/2$.