Analytic Combinatorics of Unlabeled Objects

Set of exercises 1

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1. Generating functions

Warm up formulas

Prove the following identities:

a) For every fixed $p \in \mathbb{Z}_{\geq 0}$, $[z^n]\frac{z^p}{(1-z)^{p+1}} = \binom{n}{p}$ for all $n \geq 0$.

b) $\sum_{j=0}^{n} \binom{j}{p} = \binom{n+1}{p+1}$. [Hockey-stick identity]

c) $\sum_j \binom{n}{j}\binom{m}{k-j} = \binom{n+m}{k}$. [Vandermonde's identity]

Generalized Binomial Theorem

In this exercise we prove the Generalized Binomial Theorem: for every $\alpha \in \mathbb{R}$

$$[z^n](1+z)^\alpha = \binom{\alpha}{n} := \frac{\alpha \ldots (\alpha - n + 1)}{n!}, \quad \text{for all } n \geq 0.$$

1. Fix $\alpha \in \mathbb{R}$ and define $f(z) = (1+z)^\alpha$. Show that
   a) $(1+z)f'(z) = \alpha f(z)$ is satisfied.
   b) deduce a recurrence for $f_n := [z^n]f(z)$ and prove the formula for f_n by induction in n.
2. Use the Generalized Binomial Theorem to prove that the coefficients of

$$B(z) = \frac{1 - \sqrt{1-4z}}{2z}$$

   are the Catalan numbers $[z^n]B(z) = \binom{2n}{n}\frac{1}{n+1}$.

2. Second class Stirling Numbers

Second class Stirling Numbers

The Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ count the number of partitions of a set of n elements into k non-empty subsets. Without loss of generality, we suppose the set of n elements is $[n] = \{1, \ldots, n\}$.

Prove the following identities

a) $\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k\left\{ {n-1 \atop k} \right\}$ for all[1] $n, k \geq 0$.

b) $\sum_{n\geq 0} \left\{ {n \atop k} \right\} z^n = \frac{z^k}{(1-z)(1-2z)\ldots(1-kz)}$.

[1]We define $\left\{ {n \atop k} \right\} = 0$ if every $n < 0$, $k < 0$ or $n < k$.

Second class Stirling Numbers II
In this exercise we give a combinatorial interpretation to

$$\sum_{n\geq 0} \left\{ {n \atop k} \right\} z^n = \frac{z^k}{(1-z)(1-2z)\dots(1-kz)}\,.$$

We define an algorithm. Consider a partition $P = \{S_1, \dots, S_k\}$:

- We keep an list L of the *known* parts from P. Initially $L = []$.
- We iterate $j = 1, \dots, n$. For iteration j, let $S_j \in P$ with $j \in S$. If S appears in L, write its index. If not, append it and write $|V| + 1$.

The *numbers written* belong to $[k]$. They constitute the *backbone*

$$P = \{\{4,6,7\},\{1,3\},\{2,5\}\} \mapsto L(P) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{3} & \mathbf{2} & \mathbf{3} & \mathbf{3}. \end{pmatrix}$$

Prove that this yields a bijection. Find a combinatorial specification and deduce the OGF of the partitions into k parts, k fixed.

3. Multisets and recurrences

We recall that the class $\mathcal{P}$ of integer partitions is defined by

$$\mathcal{P} = \mathtt{MSet}(\mathtt{Seq}_{\geq 1}(\mathcal{Z}))\,,$$

where $\mathtt{Seq}_{\geq 1}(\mathcal{Z})$ represents the positive integers, $\mathbf{k}$ with size k.

1. Using the OGF, prove that the number of partitions $p(n)$ of n satisfies the following recurrence for $n \geq 1$
$$np(n) = \sum_{j=1}^{n} \sigma(j) \cdot p(n-j)\,,$$
where $\sigma(j)$ is the sum of the positive divisors of j.

2. Consider $t(n)$, the number of unlabeled rooted trees with n vertices (vertices undistinguishable except root). Find a functional equation for the OGF. Derive a recurrence.