Analytic Combinatorics of Unlabeled Objects

Set of exercises III: responses

1. Prove Schur's Theorem:

   Theorem 1 (Schur's Theorem) *If c_n represents the number of representations of n as a non-negative integer combination of $a_1, \ldots, a_M$, these being a set of positive integers with $\gcd(a_1, \ldots, a_M) = 1$, then*

$$c_n \sim \frac{n^{M-1}}{(M-1)!\, a_1 \ldots a_M}\,.$$

Response. The representations correspond to multisets of $a_1, a_2, \ldots, a_M$ with sizes $|a_i| = a_i$. Thus the generating function is

$$C(z) = \sum c_n z^n = \frac{1}{1-z^{a_1}} \cdots \frac{1}{1-z^{a_M}}\,.$$

We perform partial fractions. Note that all of the roots ζ of the denominator $Q(z) = \prod_{j=1}^{M}(1-z^{a_j})$ are roots of unity with $\zeta^{a_j} = 1$ for some a_j.

The roots of each $(1-z^{a_j})$ are simple (it suffices to take the derivative), hence, the multiplicity of a root ζ of $Q(z)$ is the number of terms j such that $\zeta^{a_j} = 1$. Of course, $\zeta = 1$ has multiplicity M. All other roots have multiplicity strictly less than M. Indeed, write $\zeta = \exp(2\pi i \frac{p}{q})$. Observe that if $\zeta^{a_j} = 1$, then $q | a_j$. If this were the case for every j we would have that q is a common divisors of all a_j and hence $q = 1$.

Thus we may write

$$C(z) = \frac{A}{(1-z)^M} + \sum_{\zeta : Q(\zeta)=0} \sum_{k=1}^{M-1} \frac{\alpha_{\zeta,k}}{(1-z/\zeta)^k}\,,$$

for some coefficients A and $\alpha_{\zeta,k}$. We compute A by multiplying both sides by $(1-z)^M$ and letting $z \to 1$, to obtain $A = \frac{1}{a_1 a_2 \ldots a_M}$. T

hen we remark that

$$[z^n]\frac{\alpha_{\zeta,k}}{(1-z/\zeta)^k} = \alpha_{\zeta,k}\binom{k+n-1}{n-1}\zeta^{-n} \sim \alpha_{\zeta,k}\frac{n^{k-1}}{(k-1)!}\zeta^{-n}\,.$$

Being $|\zeta| = 1$ for all roots of $Q(z)$ and $k < M$, it follows that these coefficients are all negligible with respect to $[z^n]\frac{A}{(1-z)^M} \sim A\frac{n^{M-1}}{(M-1)!}$.

Thus

$$[z^n]C(z) \sim A\frac{n^{M-1}}{(M-1)!} = \frac{n^{M-1}}{(M-1)!\, a_1 \ldots a_M}\,.$$

Note, in particular, that all coefficients are positive for large enough n.

2. The Elias gamma code is a universal code for integers $k \geq 1$. To encode k, let $N = \lfloor \log_2 k \rfloor$, and define $C(k) = 0^N (k)_2$, where $(k)_2$ is the expansion of k in base 2.

   For example: $C(1) = 1$, $C(3) = 011$, $C(5) = 00101$, $C(20) = 000010100$.

   You can verify that no code is the prefix of another, hence C is prefix-free and uniquely decodable.

   The input. Consider $W \in \{0,1\}^{\mathbb{Z}_{\geq 0}}$ generated by iid samples of a charged-coin, which gives 0 with probability $p \geq \frac{1}{2}$ and 1 otherwise.

   The compression algorithm. We want to combine Elias coding with runlength in order to code the prefix W_n of length n of W.

   Write $W = 0^{X_1}10^{X_2}1\ldots$ where X_i are the lengths of the successive runs of 0s. Note that the X_i are iid. We write X for an arbitrary X_i. We code the prefix $W_n = 0^{X_1}1\ldots 0^{X_{R_n}}10^K$ by coding each run 0^k1 separately using $c(0^k1) = C(k+1)$ for $k \geq 0$ and, for the tail $T_n = 0^K$, we code the same as 0^K1. The receiver knows he must discard the last 1 after decoding.

   For example: $c(001 \cdot 1 \cdot 00001 \cdot 00) = 011 \cdot 1 \cdot 00101 \cdot 011$. Of course, the longer the runs, the better the compression, and this is what we want to quantify.

   The result. We are going to prove that the expected rate of compression satisfies

$$\lim_{n\to\infty} \frac{\mathbb{E}[|c(W_n)|]}{n} = (1-p)\cdot\left(1 + \frac{2}{p}\cdot\sum_{j=1}^{\infty} p^{2^j}\right).$$

   (a) Translate the decomposition into runs of 0s into an equation for

$$P(z,u) = \sum_{n,\ell\geq 0} \Pr(|c(W_n)| = \ell) z^n u^\ell,$$

   involving $A(z,u)$ and $B(z,u)$, the MGFs for a single run 0^X1 and a run of zeroes 0^K.

   (b) Deduce that $\mathbb{E}[|c(W_n)|] \sim D \times n$ where $D = \frac{\mathbb{E}[|c(0^X1)|]}{\mathbb{E}[|0^X1|]}$ is the quotient of the expected length of the coding of one run, divided by the expected length of that run.

   (c) Simplify the constant to $D = (1-p)\cdot\left(1 + \frac{2}{p}\cdot\sum_{j=1}^{\infty} p^{2^j}\right)$.

   (d) **[Extra.]** Can you prove that $\sum_{j=1}^{\infty} p^{2^j} \sim \log_2\left(\frac{1}{1-p}\right)$ as $p \to 1^-$?

Response. See Section 5.2 of `https://protondo.github.io/files/course-mpri-25/MPRI-7.pdf` which provides a more general analysis up to part (b). We now prove (c) and (d).

(c) In the current case we have $\mathbb{E}[|c(0^X1)|] = \sum_{k=1}^{\infty} p^{k-1}(1-p)(2\lfloor\log_2 k\rfloor + 1)$ since $2\lfloor\log_2 k\rfloor + 1 = |c(0^k1)|$. We have $\sum_{k=1}^{\infty} p^k \lfloor\log_2 k\rfloor = \frac{1}{1-p}\sum_{j=1}^{\infty} p^{2^j}$, because the RHS is the power-series (in p) of the partial sums:

$$[p^n]\tfrac{1}{1-p}\sum_{j=1}^{\infty} p^{2^j} = \sum_{j:2^j\leq n} 1 = \lfloor\log_2 n\rfloor.$$

Thus $\mathbb{E}[|c(0^X 1)|] = 1 + \frac{1}{p}\sum_{j=1}^{\infty} p^{2^j}$. On the other hand, $\mathbb{E}[|0^X 1|] = \sum_{k=1}^{\infty} p^{k-1}(1-p)k = \frac{1}{1-p}$. Thus the formula follows.

(d) We begin from $\frac{1}{1-p}\sum_{j=1}^{\infty} p^{2^j} = \sum_{k=1}^{\infty} p^k \lfloor \log_2 k \rfloor$. Observe that $\lfloor \log_2 k \rfloor = \log_2 k + O(1)$, thus $\sum_{k=1}^{\infty} p^k \lfloor \log_2 k \rfloor = \sum_{k=1}^{\infty} p^k \log_2 k + O(\frac{1}{1-p})$.

We recall that the harmonic numbers $H_k = \sum_{j=1}^{n} \frac{1}{j}$ satisfy $H_k = \log k + O(1)$. Thus we have $\sum_{k=1}^{\infty} p^k \lfloor \log_2 k \rfloor = \frac{1}{\log 2}\sum_{k=1}^{\infty} p^k H_k + O(\frac{1}{1-p})$. We recall[1] that $\sum_{k=1}^{\infty} p^k H_k = \frac{1}{1-p}\log\left(\frac{1}{1-p}\right)$ for $|p| < 1$. Thus we have

$$\frac{1}{1-p}\sum_{j=1}^{\infty} p^{2^j} = \sum_{k=1}^{\infty} p^k \lfloor \log_2 k \rfloor = \frac{1}{\log 2}\frac{1}{1-p}\log\left(\frac{1}{1-p}\right) + O(\tfrac{1}{1-p}),$$

and the result follows.

3. In this exercise we use complex-integration to prove that

$$\sum_{n=0}^{\infty} \binom{2n}{n} z^n = \frac{1}{\sqrt{1-4z}},$$

for $|z| < 1/4$.

(a) Prove that $F(z) \triangleq \sum_{n=0}^{\infty} \binom{2n}{n} z^n$ converges for $|z| < \frac{1}{4}$.

(b) Fix $r > 0$. Remark that $\oint_{|u|=r} \frac{(1+u)^{2n}}{u^{n+1}} du = 2\pi \binom{2n}{n}$. Use this to prove $F(z) = \frac{1}{\sqrt{1-4z}}$.

(c) Deduce that $\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{n+1} z^n = \frac{1-\sqrt{1-4z}}{2z}$.

Responses.

(a) We note that $\binom{2n}{n} \le \sum_j \binom{2n}{j} = 4^n$, thus the coefficients are dominated by those of $|4^n z^n|$ and the series converges absolutely for $|z| < \frac{1}{4}$.

(b) The first identity follows from the Binomial Theorem, $[u^n](1+u)^{2n} = \binom{2n}{n}$ and Cauchy's Formula applied for the circle (the simplest case, which could be computed directly).

Consider $|z| < 1/4$ and let $r = 1$. Then $\frac{(1+r)^2}{r}|z| = 4|z| < 1$. Thus $|\frac{(1+u)^2}{u} z| \le \frac{(1+r)^2}{r}|z| < 1$ and the series converges $\sum \left(\frac{(1+u)^2}{u} z\right)^n$ converges uniformly. Integrating over $|u| = r = 1$, by the uniform convergence we may exchange sums and integrals:

$$\begin{aligned}\sum_{n=0}^{\infty} \binom{2n}{n} z^n &= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\left(\oint_{|u|=1} \frac{(1+u)^{2n}}{u^{n+1}} du\right) z^n \\ &= \frac{1}{2\pi i}\oint_{|u|=1}\left(\sum_{n=0}^{\infty}\left(\frac{(1+u)^2}{u} z\right)^n\right)\frac{du}{u} \\ &= \frac{1}{2\pi i}\oint_{|u|=1} \frac{1}{1-\frac{(1+u)^2}{u} z}\frac{du}{u}.\end{aligned}$$

[1] Recall that $\log\frac{1}{1-x} = \sum \frac{1}{k} z^k$ and so, for the partial sums, we have $\frac{1}{1-x}\log\frac{1}{1-x} = \sum H_k z^k$.

Now it all comes down to compute the latter integral. Cleaning out the denominators, we want to integrate $u \mapsto \frac{1}{u-(1+u)^2 z}$ over the circle $|u| = 1$. The denominator has two potential roots:

$$u_\pm = u_\pm(z) = \frac{1 - 2z \pm \sqrt{1-4z}}{2z}\,.$$

We remark, by the Newton relations for the roots of the polynomials, that $u_+ \times u_- = 1$, because these are roots of $u^2 + (2 - 1/z)u + 1 = 0$. Hence it follows that $|u_+| \times |u_-| = 1$. Moreover $u_+ + u_- = 1/z - 2$, whence $|u_+| + |u_-| \geq 1/|z| - 2 > 2$ and it follows that at least one of the roots satisfies $|u| > 1$. By $|u_+| \times |u_-| = 1$, the other root must satisfy $|u| < 1$. Observe that $|u_+| \to \infty$ as $z \to 0$, thus, by continuity of both roots, we deduce that $|u_+| > 1$ and $|u_-| < 1$ for every $|z| < 1/4$.

Write

$$\frac{1}{u-(1+u)^2 z} = \frac{A}{u-u_-} + \frac{B}{u-u_+}\,,$$

where $A = A(z)$ and $B = B(z)$ actually depend on z.

By Cauchy's Formula, since only u_- lies within the circle $|u| = 1$, we deduce

$$\oint_{|u|=1} \frac{du}{u-(1+u)^2 z} = 2\pi i A\,, \qquad A = \lim_{u \to u_-} \frac{u-u_-}{u-(1+u)^2 z} = \frac{1}{(u-(1+u)^2 z)'|_{u=u_-}}\,.$$

And we can verify that $A = \frac{1}{\sqrt{1-4z}}$. Thus the formula follows.

(c) This part follows by noticing that, by integration of the series

$$\sum_{n=0}^{\infty} \binom{2n}{n} \tfrac{1}{n+1} z^n = \tfrac{1}{z} \int_0^z \left(\sum_{n=0}^{\infty} \binom{2n}{n} u^n \right) du\,,$$

which is valid for $|z| < 1/4$, within the radius of convergence.

Since

$$\int_0^z \left(\sum_{n=0}^{\infty} \binom{2n}{n} u^n \right) du = \int_0^z \frac{du}{\sqrt{1-4u}} = \frac{1 - \sqrt{1-4z}}{2}\,,$$

the result follows.