Analytic Combinatorics of Unlabeled Objects

Set of exercises I: responses

1. Using generating functions prove the following:

   (a) $\sum_{j=0}^{n} \binom{j}{p} = \binom{n+1}{p+1}$. [Hockey-stick identity]

   (b) $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$.

   (c) $\sum_{j=1}^{n} j^p \sim n^{p+1}/(p+1)$ for every integer $p \geq 0$.

   (d) $\sum_j \binom{n}{j}\binom{m}{k-j} = \binom{n+m}{k}$. [Vandermonde's identity]

Responses.

(a) Recalling that if $A(z) = \sum a_n z^n$ we have $\frac{1}{1-z}A(z) = \sum_n (\sum_{j \leq n} a_j) z^n$, the left-hand side is the application to $A(z) = \frac{z^p}{(1-z)^{p+1}}$, the OGF of $(\binom{n}{p})_n$.

Thus we obtain $[z^n]\frac{1}{1-z}\frac{z^p}{(1-z)^{p+1}} = [z^n]\frac{z^p}{(1-z)^{p+2}} = [z^{n+1}]\frac{z^{p+1}}{(1-z)^{p+2}} = \binom{n+1}{p+1}$.

(b) We recall that if $A(z) = \sum a_n z^n$, then $z\partial_z A(z) = \sum a_n n z^n$. Thus, applying this twice to $\frac{1}{1-z} = \sum z^n$, we have

$$\sum_{n=0}^{\infty} n^2 z^n = (z\partial_z)^2 \tfrac{1}{1-z} = \frac{z+z^2}{(1-z)^3} .$$

Now, by the same argument of (a), the OGF of the partial sums is:

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} j^2 \right) z^n = \frac{z+z^2}{(1-z)^4} .$$

Now we remark that $[z^n]\frac{z}{(1-z)^4} = [z^{n+2}]\frac{z^3}{(1-z)^4} = \binom{n+2}{3}$ and $[z^n]\frac{z^2}{(1-z)^4} = [z^{n+1}]\frac{z^3}{(1-z)^4} = \binom{n+1}{3}$. We obtain $\sum_{j=0}^{n} j^2 = \binom{n+2}{3} + \binom{n+1}{3}$.

Optionally, one can also do: $j^2 = 2\binom{j}{2} + \binom{j}{1}$ and then use $\sum \binom{n}{p} z^n = \frac{z^p}{(1-z)^{p+1}}$.

(c) We proceed by induction. This is clearly true for $p = 0$. Now assume this to hold for all $p < q$. We show this for q.

Notice that $\binom{j}{q} = \frac{j(j-1)\ldots(j-q+1)}{q!} = \frac{j^q}{q!} + O(j^{q-1})$. Summing over j we have, by part (a),

$$\binom{n+1}{q+1} = \sum_{j=0}^{n} \binom{j}{q} = \tfrac{1}{q!} \sum_{j=0}^{n} j^q + O(\sum_{j=0}^{n} j^{q-1}) .$$

By the inductive hypothesis $\sum_{j=0}^{n} j^{q-1} \sim n^q/q$. Thus the remainder is actually $O(n^q)$ and we deduce

$$\binom{n+1}{q+1} = \sum_{j=0}^{n} \binom{j}{q} = \tfrac{1}{q!} \sum_{j=0}^{n} j^q + O(n^q) .$$

Since $\binom{n+1}{q+1} = \frac{n^{q+1}}{(q+1)!} + O(n^q)$ by the same argument, we obtain

$$\sum_{j=0}^{n} j^q = \frac{q!}{(q+1)!} n^{q+1} + O(n^q) = \frac{1}{q+1} n^{q+1} + O(n^q)\,.$$

(d) Multiply through by z^k and sum over k. We use the usual convention (in order to avoid the limits of summation) that we sum over $k \in \mathbb{Z}$, $j \in \mathbb{Z}$. Note that $\binom{a}{b} = 0$ for $b > a$ or $b < 0$. Thus the expressions make sense.

We obtain:

$$\begin{aligned}
\sum_k \left(\sum_j \binom{n}{j} \binom{m}{k-j} \right) z^k &= \sum_j \binom{n}{j} \sum_k \binom{m}{k-j} z^k \\
&= \sum_j \binom{n}{j} z^j \sum_k \binom{m}{k-j} z^{k-j} \\
&= \sum_j \binom{n}{j} z^j (1+z)^m \\
&= (1+z)^{n+m} \\
&= \sum_k \binom{n+m}{k} z^k\,.
\end{aligned}$$

2. Consider the question:

   Given an OGF $F(z) = \sum_{n\geq 0} a(n)\, z^n$, and $q \in \mathbb{Z}_{\geq 1}$
   how to obtain an OGF for $\sum_{n \geq 0} a(n\,q)\, z^{n\,q}$?

(a) Let $\omega = \exp(2\pi i/q)$. Prove that

$$\sum_{n\geq 0} a(n\,q)\, z^{n\,q} = \frac{1}{q} \sum_{k=0}^{q-1} F(z\omega^k)\,. \tag{1}$$

(b) Using (1), prove that if F has radius of convergence R_F, for $0 \leq c < R_F$,

$$a(0) = \int_0^1 F(ce^{2\pi i t}) dt\,.$$

(c) Obtain a formula for $\sum_{n\geq 0} a(n\,q + r)\, z^{n\,q+r}$ with $r \in \{0, \ldots, q-1\}$.

Responses.

(a) We remark that

$$\begin{aligned}
\sum_{k=0}^{q-1} F(z\omega^k) &= \sum_{k=0}^{q-1} \sum_n a_n \omega^{kn} z^n \\
&= \sum_n a_n z^n \left(\sum_{k=0}^{q-1} \omega^{kn} \right)\,.
\end{aligned}$$

Here, by the Geometric sum,

$$\sum_{k=0}^{q-1} \omega^{kn} = \sum_{k=0}^{q-1} (\omega^n)^k = \begin{cases} \frac{1-(\omega^n)^q}{1-\omega^n}, & \text{if } \omega^n \neq 1\,, \\ q\,, & \text{if } \omega^n = 1\,. \end{cases}$$

Being ω a q root of unity, $(\omega^n)^q = 1$. On the other hand, $\omega^n = 1$ iff $n \equiv 1 (\mathrm{mod} q)$. Therefore we deduce that $\sum_{k=0}^{q-1} \omega^{kn} = q \times \mathbf{1}_{n \equiv 0 \bmod q}$, and part (a) follows.

(b) The right-hand side of (1) is actually a Riemann sum:

$$\tfrac{1}{q} \sum_{k=0}^{q-1} F(z\omega^k) = \tfrac{1}{q} \sum_{k=0}^{q-1} F(z \exp(2\pi i k/q))\,.$$

If $z = c < R_F$, within its radius of convergence F is continuous and therefore $t \mapsto F(c\exp(2\pi i t))$ is Riemann-integrable. Taking $q \to \infty$ we obtain $\frac{1}{q}\sum_{k=0}^{q-1} F(c\omega^k) \to \int_0^1 F(ce^{2\pi i t})dt$.

Now, from part (a) we remark that $\frac{1}{q}\sum_{k=0}^{q-1} F(c\omega^k) = a(0) + a(q)c^q + \ldots$. When $q \to \infty$ we have that $a(q)c^q + a(2q)c^{2q} + \ldots$ is bounded by the tail of a convergence series $\sum_n a(n)c^n$, thus it tends to 0. We deduce $\frac{1}{q}\sum_{k=0}^{q-1} F(c\omega^k) \to a(0)$.

(c) For $r > 0$, consider $G(z) = F(z)z^{q-r}$ and apply (a) to $G(z)$.

3. The Stirling numbers of the second kind $\left\{ n \atop k \right\}$ count the number of partitions of a set of n elements into k non-empty subsets. Without loss of generality, we suppose the set of n elements is $[n] = \{1, \ldots, n\}$.

Prove the following identities

(a) $\left\{ n \atop k \right\} = \left\{ n-1 \atop k-1 \right\} + k\left\{ n-1 \atop k \right\}$ for all[1] $n, k \geq 0$.

(b) $\sum_{n\geq 0} \left\{ n \atop k \right\} z^n = \frac{z^k}{(1-z)(1-2z)\ldots(1-kz)}$.

Find a formula for $\left\{ n \atop k \right\}$ by applying partial fractions.

Response.

(a) In a partition of $[n] = \{1, \ldots, n\}$ into k parts, either n is alone making its own part, or not.

– The total number of partitions in which n is alone is $\left\{ n-1 \atop k-1 \right\}$, as the rest is a partition of $[n-1]$ into $k-1$ parts.

– The total number of partitions in which n is not alone is $k \times \left\{ n-1 \atop k \right\}$. Indeed, erasing n we have a partition of $[n-1]$ into k parts. Then we must decide to which of these k parts n belongs to, hence the factor k.

[1]We define $\left\{ n \atop k \right\} = 0$ if every $n < 0$, $k < 0$ or $n < k$.

(b) Multiply both sides of the recurrence by k and sum over $k \in \mathbb{Z}$, with the convention $\left\{{a \atop b}\right\} = 0$ if $a < b$ or $a < 0$. We obtain

$$\begin{aligned}
\sum_n \left\{{n \atop k}\right\} z^n &= \sum_n \left\{{n-1 \atop k-1}\right\} z^n + k \sum_n \left\{{n-1 \atop k}\right\} z^n \\
&= z \sum_n \left\{{n-1 \atop k-1}\right\} z^{n-1} + kz \sum_n \left\{{n-1 \atop k}\right\} z^{n-1} .
\end{aligned}$$

Letting $A_k(z) = \sum_n \left\{{n \atop k}\right\} z^k$, we obtain

$$A_k(z) = zA_{k-1}(z) + kzA_k(z) \Longrightarrow A_k(z) = \frac{z}{1-kz} A_{k-1}(z) .$$

Iterating until $k = 0$, $A_0(z) = 1$ and we have

$$A_k(z) = \frac{z}{1-z} \frac{z}{1-2z} \cdots \frac{z}{1-kz} ,$$

as desired.

Now we apply partial fractions:

$$F_k(z) = \frac{1}{1-z} \frac{1}{1-2z} \cdots \frac{1}{1-kz} = \sum_{j=1}^{k} \frac{\alpha_j}{1-jz} ,$$

for some coefficients α_j. We compute α_j by multiplying through by $1-jz$ and making $j \to 1/z$

$$\begin{aligned}
\alpha_j &= \lim_{z \to 1/j} (1-jz) F_k(z) \\
&= \prod_{r:1 \le r \le k, r \ne j} \frac{1}{1-r/j} \\
&= j^{k-1} \prod_{r=1}^{j-1} \frac{1}{j-r} \times \prod_{r=j+1}^{k} \frac{1}{r-j} \\
&= (-1)^{k-j} j^{k-1} \frac{1}{(j-1)!(k-j)!} \\
&= (-1)^{k-j} j^{k} \frac{1}{j!(k-j)!} \\
&= (-1)^{k-j} j^{k} \frac{1}{j!(k-j)!} .
\end{aligned}$$

Thus

$$\begin{aligned}
\left\{ {n \atop k} \right\} &= [z^n] A_k(z) \\
&= [z^{n-k}] \frac{1}{(1-z)\dots(1-kz)} \\
&= \sum_{j=1}^{k} \alpha_j [z^{n-k}] \frac{1}{1-jz} = \sum_{j=1}^{k} \alpha_j j^{n-k} \\
&= \sum_{j=1}^{k} (-1)^{k-j} \frac{j^n}{j!(k-j)!} \,.
\end{aligned}$$

4. In this exercise we give a combinatorial interpretation to

$$\sum_{n \geq 0} \left\{ {n \atop k} \right\} z^n = \frac{z^k}{(1-z)(1-2z)\dots(1-kz)} \,.$$

We define an algorithm. Consider a partition $P = \{S_1, \dots, S_k\}$:

- We keep an list L of the *known* parts from P. Initially $L = []$.
- We iterate $j = 1, \dots, n$. For iteration j, let $S_j \in P$ with $j \in S_j$. If S_j appears in L, write its index. If not, append it and write $|V| + 1$.

The *numbers written* belong to $[k]$. They constitute the *backbone*

$$P = \{\{4,6,7\}, \{1,3\}, \{2,5\}\} \mapsto L(P) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{3} & \mathbf{2} & \mathbf{3} & \mathbf{3}. \end{pmatrix}$$

Prove that this yields a bijection. Find a combinatorial specification and deduce the OGF of the partitions into k parts, k fixed.

Response. Essentially, reading the numbers in $L(P)$ gives back the partition. We remark that we have numbered the elements of the partition according to the order given by the minimum element in each part. A valid backbone $L(P)$, then, must have the first appearance of j before the appearance of $j+1$, for each $j \in [k]$. Given any sequence with this properties, it corresponds to a unique partition and vice-versa. Thus we obtain a bijection. Put another way, the numbers in $L(P)$ label the parts of the partition by their order of discovery when reading $j = 1, 2, \dots, n$.

We note that $L(P)$ can be specified as

$$\{1\} \times \mathtt{Seq}(\{1\}) \times \{2\} \times \mathtt{Seq}(\{1,2\}) \times \dots \times \{k\} \times \mathtt{Seq}(\{1, \dots, k\}) \,,$$

where each number has weight 1.

The corresponding OGF is precisely

$$\frac{z^k}{(1-z)(1-2z)\dots(1-kz)} \,,$$

as desired.