Analytic Combinatorics of Unlabeled Objects

Set of exercises 3

October 6, 2025

1. Rational asymptotics

Prove Schur's Theorem:

Theorem 1 (Schur's Theorem)**.** *If c_n represents the number of representations of n as a non-negative integer combination of $a_1, \dots, a_M$, these being a set of positive integers with $\gcd(a_1, \dots, a_M) = 1$, then*

$$c_n \sim \frac{n^{M-1}}{(M-1)!\, a_1 \dots a_M}\,.$$

2. Analysis of a run-length encoding

Elias gamma code. The Elias gamma code is a universal code for integers $k \geq 1$. To encode k, let $N = \lfloor \log_2 k \rfloor$, and define $C(k) = 0^N (k)_2$, where $(k)_2$ is the expansion of k in base 2.
For example: $C(1) = 1$, $C(3) = 011$, $C(5) = 00101$, $C(20) = 000010100$.
You can verify that no code is the prefix of another, hence C is prefix-free and uniquely decodable.

The input. Consider $W \in \{0,1\}^{\mathbb{Z}_{\geq 0}}$ generated by iid samples of a charged-coin, which gives 0 with probability $p \geq \frac{1}{2}$ and 1 otherwise.

The compression algorithm. We want to combine Elias coding with runlength in order to code the prefix W_n of length n of W.

Write $W = 0^{X_1} 1 0^{X_2} 1 \dots$ where X_i are the lengths of the successive runs of 0s. Note that the X_i are iid. We write X for an arbitrary X_i. We code the prefix $W_n = 0^{X_1} 1 \dots 0^{X_{R_n}} 1 0^K$ by coding each run $0^k 1$ separately using $c(0^k 1) = C(k+1)$ for $k \geq 0$ and, for the tail $T_n = 0^K$, we code the same as $0^K 1$. The receiver knows he must discard the last 1 after decoding.

For example: $c(001 \cdot 1 \cdot 00001 \cdot 00) = 011 \cdot 1 \cdot 00101 \cdot 011$. Of course, the longer the runs, the better the compression, and this is what we want to quantify.

The result. We are going to prove that the expected rate of compression satisfies

$$\lim_{n \to \infty} \frac{\mathbb{E}[|c(W_n)|]}{n} = (1-p) \cdot \left(1 + \frac{2}{p} \cdot \sum_{j=1}^{\infty} p^{2^j}\right).$$

1. Translate the decomposition into runs of 0s into an equation for

$$P(z,u) = \sum_{n,\ell \geq 0} \Pr(|c(W_n)| = \ell) z^n u^\ell\,,$$

   involving $A(z,u)$ and $B(z,u)$, the MGFs for a single run $0^X 1$ and a run of zeroes 0^K.

2. Deduce that $\mathbb{E}[|c(W_n)|] \sim D \times n$ where $D = \frac{\mathbb{E}[|c(0^X 1)|]}{\mathbb{E}[|0^X 1|]}$ is the quotient of the expected length of the coding of one run, divided by the expected length of that run.

3. Simplify the constant to $D = (1-p)\cdot\left(1 + \frac{2}{p}\cdot\sum_{j=1}^{\infty} p^{2^j}\right)$.

4. **[Extra.]** Can you prove that $\sum_{j=1}^{\infty} p^{2^j} \sim \log_2\left(\frac{1}{1-p}\right)$ as $p \to 1^-$?

Note. In fact, by looking at the second moment it is possible to prove that $|c(W_n)|/n$ actually converges in probability to D.

3. The generating function of the central coefficient

In this exercise we use complex-integration to prove that

$$\sum_{n=0}^{\infty} \binom{2n}{n} z^n = \frac{1}{\sqrt{1-4z}} ,$$

for $|z| < 1/4$.

1. Prove that $F(z) \triangleq \sum_{n=0}^{\infty} \binom{2n}{n} z^n$ converges for $|z| < \frac{1}{4}$.

2. Fix $r > 0$. Remark that $\oint_{|u|=r} \frac{(1+u)^{2n}}{u^{n+1}} du = 2\pi\binom{2n}{n}$. Use this to prove $F(z) = \frac{1}{\sqrt{1-4z}}$.

3. Deduce that $\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{n+1} z^n = \frac{1-\sqrt{1-4z}}{2z}$.