Analytic Combinatorics of Unlabeled Objects

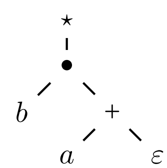
Set of exercises 2

September 29, 2025

1. Regular expressions

Consider the combinatorial class $\mathcal{R}$ of regular expressions as unary-binary trees:

- leaves are decorated by the letters "a", "b" or the empty word "ε".
- internal binary nodes are decorated by the operators union "+" or concatenation "•",
- internal unary-nodes are decorated by the Kleene-star "⋆".



$$(b \cdot (a + \varepsilon))^\star$$

Give a combinatorial specification. Determine the ordinary generating function supposing that the size is the total number of nodes.

2. Rational asymptotics

Prove Schur's Theorem:

Theorem 1 (Schur's Theorem)**.** *If c_n represents the number of representations of n as a non-negative integer combination of $a_1, \ldots, a_M$, these being a set of positive integers with $\gcd(a_1, \ldots, a_M) = 1$, then*

$$c_n \sim \frac{n^{M-1}}{(M-1)!\, a_1 \ldots a_M}\,.$$

3. Multivariate Generating Functions

A) Integer compositions

We recall that integer compositions are specified by

$$\mathcal{C} = \texttt{Seq}(\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots\}) \simeq \texttt{Seq}(\texttt{Seq}_{\geq 1}(\mathcal{Z}))\,.$$

During class, we used OGFs to prove that the average number of terms in a composition of n is $\sim n/2$.

1. What if we restricted the compositions to have an even number of terms? Find the total number of compositions of n into an even number of terms.

2. Let us consider now the number of even terms in a general composition of n. Show that the average number of even terms is $\sim n/6$.

3. Now we want to fix the number m of even terms. Find the generating function.

B) Binary trees

We consider the binary trees

$$\mathcal{B} = \mathcal{E} + \mathcal{Z} \times \mathcal{B} \times \mathcal{B}\,,$$

where the node $\mathcal{E}$ is the empty-node, while $\mathcal{Z}$ is an atom of weight 1.

1. Give a specification for the binary trees $\mathcal{B}^\circ$ marking the length of the left-most branch.

2. Solve the equation for $B(z,u)$ and show $\partial_u B(z,1) = z(B(z))^3$.

3. Let $A(z) = zB(z)$. Apply the Lagrange Inversion Formula to compute $[z^n](A(z))^3$.

4. Deduce that the average length of the left-most branch[1] tends to 3 as $n \to \infty$.

5. Generalize. Find the exact number of binary trees of size n with left-most branch of length m.

[1] We will later see in class that the average-depth of a random binary tree is rather $\Theta(\sqrt{n})$.