Analytic Combinatorics of Unlabeled Objects

Set of exercises 1

September 22, 2025

1. Generating functions

Exercise 1
Using generating functions prove the following:

a) $\sum_{j=0}^{n} \binom{j}{p} = \binom{n+1}{p+1}$. [Hockey-stick identity]

b) $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$.

c) $\sum_{j=1}^{n} j^p \sim n^{p+1}/(p+1)$ for every integer $p \geq 0$.

d) $\sum_j \binom{n}{j}\binom{m}{k-j} = \binom{n+m}{k}$. [Vandermonde's identity]

2. Arithmetic progressions in a generating function

Question
Given an OGF $F(z) = \sum_{n\geq 0} a(n)\, z^n$, and $q \in \mathbb{Z}_{\geq 1}$ how to obtain an OGF for $\sum_{n\geq 0} a(n\, q)\, z^{n\, q}$?

a) Let $\omega = \exp(2\pi i/q)$. Prove that

$$\sum_{n\geq 0} a(n\, q)\, z^{n\, q} = \frac{1}{q}\sum_{k=0}^{q-1} F(z\omega^k)\,. \qquad (1)$$

b) Using (1), prove that if F has radius of convergence R_F, for $0 \leq c < R_F$,

$$a(0) = \int_0^1 F(ce^{2\pi i t})dt\,.$$

c) Obtain a formula for $\sum_{n\geq 0} a(n\, q + r)\, z^{n\, q + r}$ with $r \in \{0, \ldots, q-1\}$.

3. Second class Stirling Numbers

Second class Stirling Numbers
The Stirling numbers of the second kind $\left\{ n \atop k \right\}$ count the number of partitions of a set of n elements into k non-empty subsets. Without loss of generality, we suppose the set of n elements is $[n] = \{1, \dots, n\}$.

Prove the following identities

a) $\left\{ n \atop k \right\} = \left\{ n-1 \atop k-1 \right\} + k\left\{ n-1 \atop k \right\}$ for all[1] $n, k \geq 0$.

b) $\sum_{n\geq 0} \left\{ n \atop k \right\} z^n = \frac{z^k}{(1-z)(1-2z)\dots(1-kz)}$.

Find a formula for $\left\{ n \atop k \right\}$ by applying partial fractions.

Second class Stirling Numbers II
In this exercise we give a combinatorial interpretation to

$$\sum_{n\geq 0} \left\{ n \atop k \right\} z^n = \frac{z^k}{(1-z)(1-2z)\dots(1-kz)}.$$

We define an algorithm. Consider a partition $P = \{S_1, \dots, S_k\}$:

- We keep an list L of the *known* parts from P. Initially $L = []$.
- We iterate $j = 1, \dots, n$. For iteration j, let $S_j \in P$ with $j \in S$. If S appears in L, write its index. If not, append it and write $|V| + 1$.

The *numbers written* belong to $[k]$. They constitute the *backbone*

$$P = \{\{4,6,7\},\{1,3\},\{2,5\}\} \mapsto L(P) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{3} & \mathbf{2} & \mathbf{3} & \mathbf{3}. \end{pmatrix}$$

Prove that this yields a bijection. Find a combinatorial specification and deduce the OGF of the partitions into k parts, k fixed.

4. Multisets and recurrences

We recall that the class $\mathcal{P}$ of integer partitions is defined by

$$\mathcal{P} = \mathtt{MSet}(\mathtt{Seq}_{\geq 1}(\mathcal{Z})),$$

where $\mathtt{Seq}_{\geq 1}(\mathcal{Z})$ represents the positive integers, $\mathbf{k}$ with size k.

1. Using the OGF, prove that the number of partitions $p(n)$ of n satisfies the following recurrence for $n \geq 1$
$$np(n) = \sum_{j=1}^{n} \sigma(j) \cdot p(n-j),$$
where $\sigma(j)$ is the sum of the positive divisors of j.

2. Consider $t(n)$, the number of unlabeled rooted trees with n vertices (vertices undistinguishable except root). Find a functional equation for the OGF. Derive a recurrence.

[1] We define $\left\{ n \atop k \right\} = 0$ if every $n < 0$, $k < 0$ or $n < k$.