Semantic reductions in random expression trees

Pablo Rotondo

LIGM, Université Gustave Eiffel

Based on joint work with
Florent Koechlin, Cyril Nicaud, Carine Pivoteau

Journée mathématiques informatique,
Rouen, 12 December, 2025.

Plan of the talk

1. Random expression trees: BST and uniform
2. Semantic simplifications
3. Results for uniform random expression trees
4. Result for BST-like trees and elements of the proof
5. Conclusions

Section

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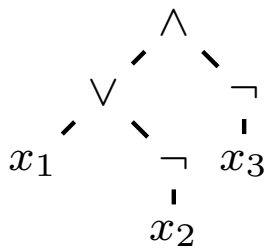
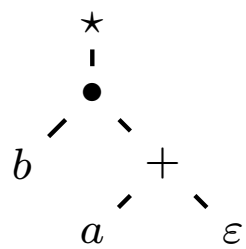
- Expression trees



$(x_1 \vee \neg x_2) \wedge \neg x_3$



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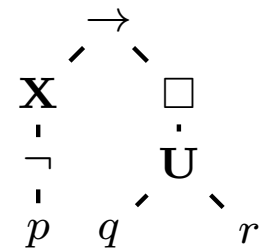

$\mathbf{X}(\neg p) \rightarrow \square(q \mathbf{U} r)$

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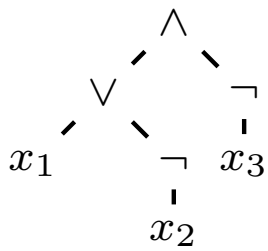

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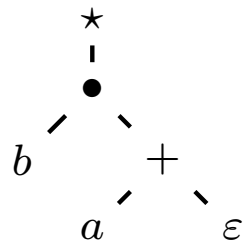

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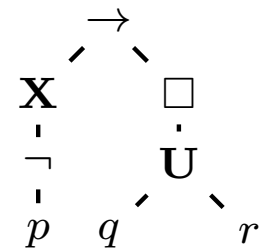

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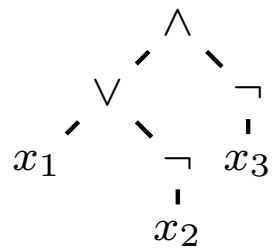

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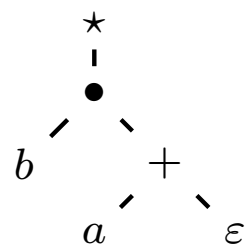

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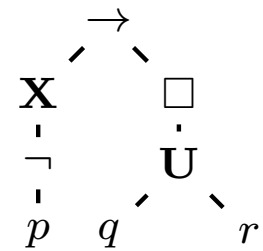

$\mathbf{X}(\neg p) \rightarrow \Box(q \mathbf{U} r)$

- Automated testing, benchmark testing
  - Correctness and performance of algorithms
- Randomly generated input
  - Realistic distribution
  - Simple implementation, possibility of theoretical analysis.

BST-like trees: a natural construction algorithm

Problem: sample random tree of size n,

- representing expression with unary and binary operators,
- leaves correspond to constants or variables.

[1] If $n = 2$, constrained to arity one.

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Example.

$$\Pr_n\left(\begin{array}{ccc} & \star & \\ & | & \\ & + & \\ a \swarrow & & \searrow \star \\ & & | \\ & & b \end{array}\right) = p_\star p_+ \tfrac{1}{2} p_a p_b$$

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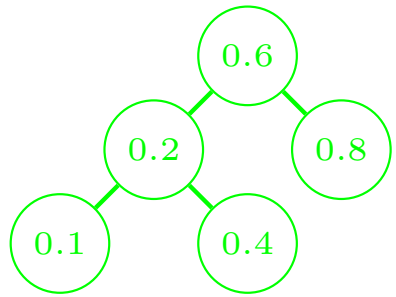
Why binary search "like" ?

- Build BST from n random numbers $u_i \in [0, 1]$:



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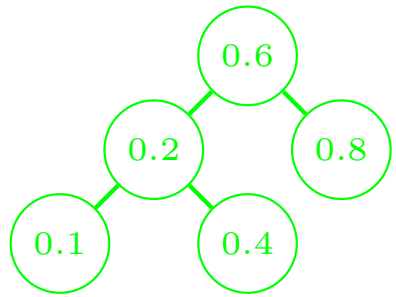

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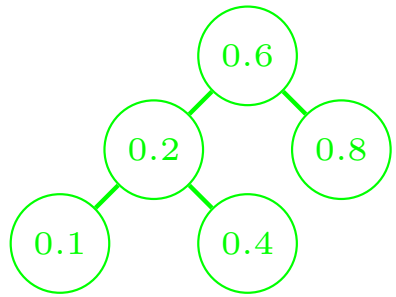

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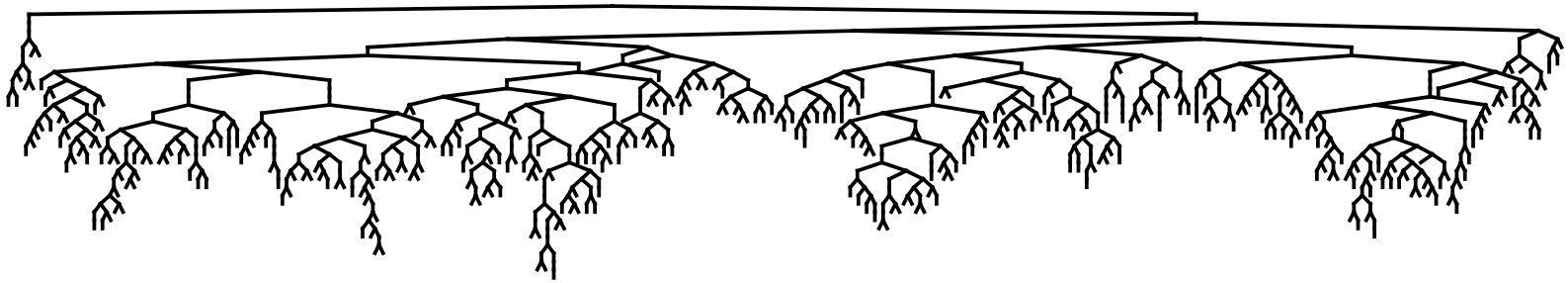
Same construction: force our subtrees to have $|T_L|, |T_R| \geq 1$, as node corresponds to binary operator.

BST-like trees: a natural construction algorithm

Code used in tool `lbtt` (from `TCS`) to draw an LTL formula:

function RandomFormula(n):
if $n = 1$ **then**
  $p :=$ random symbol in $AP \cup \{\top, \bot\}$;
  return p;
else if $n = 2$ **then**
  $op :=$ random unary operator in $\{\neg, \mathbf{X}, \Box, \Diamond\}$;
  $f :=$ RandomFormula(1);
  return $op\ f$;
else
  $op :=$ random operator in $\{\neg, \mathbf{X}, \Box, \Diamond, \wedge, \vee, \rightarrow, \leftrightarrow, \mathbf{U}, \mathbf{R}\}$;
  if op *in* $\{\neg, \mathbf{X}, \Box, \Diamond\}$ **then**
    $f :=$ RandomFormula($n - 1$);
    return $op\ f$;
  else
    $x :=$ uniform integer in $[1, n - 2]$;
    $f_1 :=$ RandomFormula(x);
    $f_2 :=$ RandomFormula($n - x - 1$);
    return $(f_1\ op\ f_2)$;

BST-like trees: distribution over unary-binary trees



BST-like tree distribution is not uniform[a].

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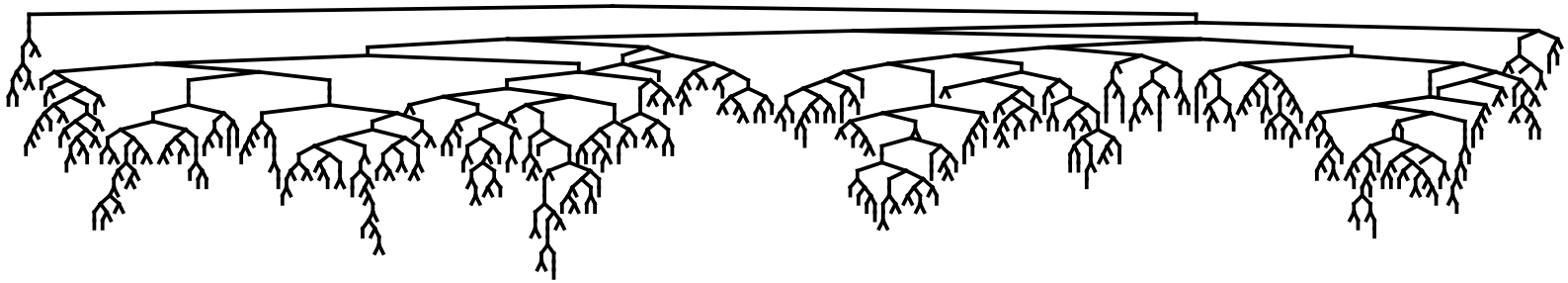


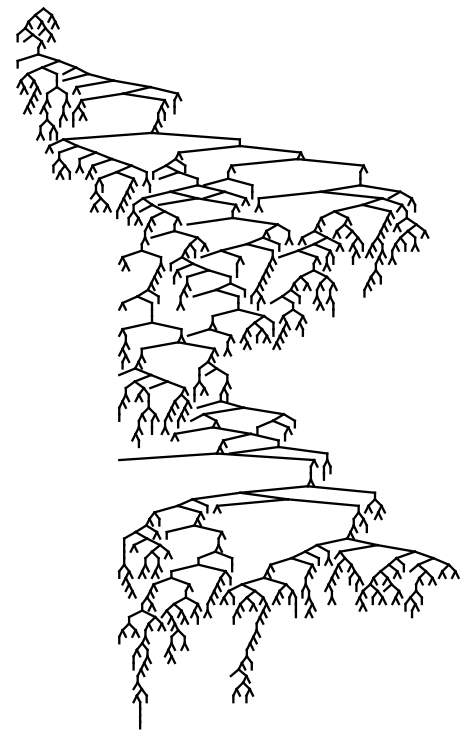
BST-like tree distribution is not uniform[a].

- Binary nodes $\approx$ balanced $\frac{n}{2}$–$\frac{n}{2}$,
  but for uniform trees

$$\mathbb{E}_n[\min(|T_L|, |T_R|)] \sim c_0\sqrt{n}\,.$$

- Expected height of different order

$$\Theta(\log n) \text{ vs } \Theta(\sqrt{n})\,.$$



[a] Tree T chosen uniformly from $\{T : |T| = n\}$.

Uniform and BST-like distributions

The uniform distribution:

- naturally maximizes entropy.
- can be sampled efficiently with a little more effort
  (Recursive method, Boltzmann samplers, Devroye's constrained GW).
- is amenable to theoretical study (Analytic Combinatorics).

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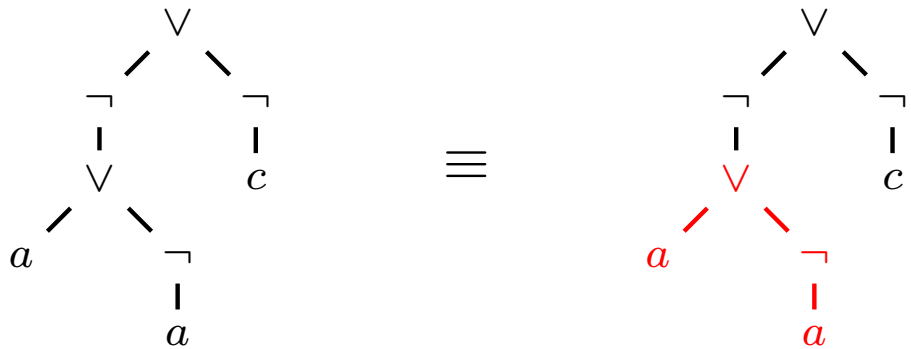
We had previously studied *semantically* **uniform expressions**...

Section

Semantic simplification

Given tree may be redundant



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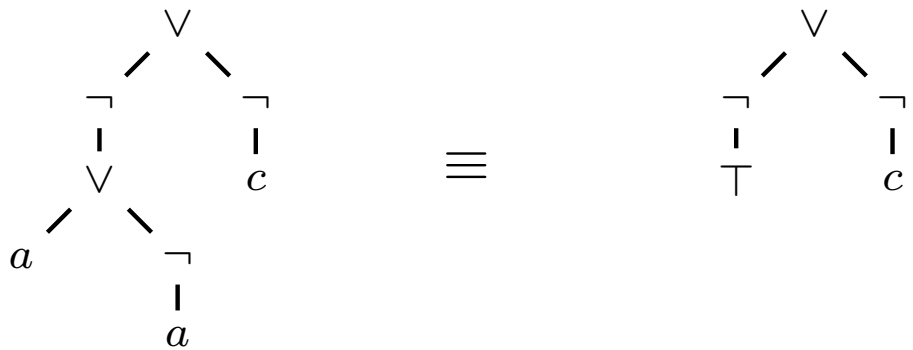

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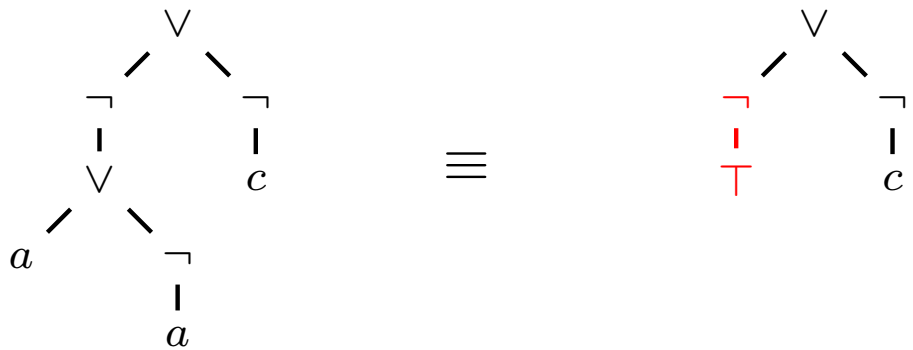

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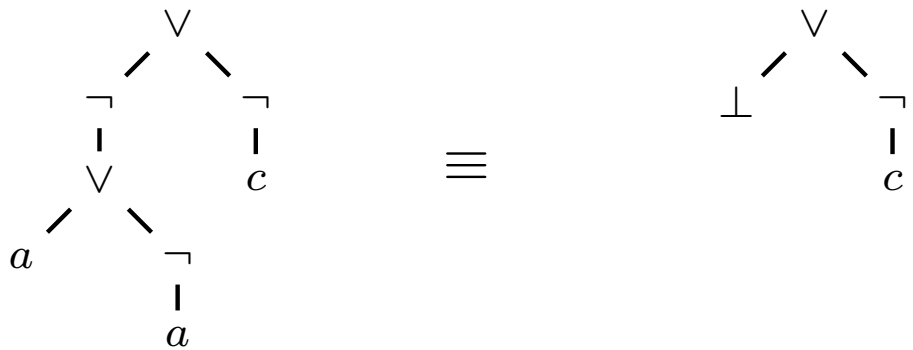

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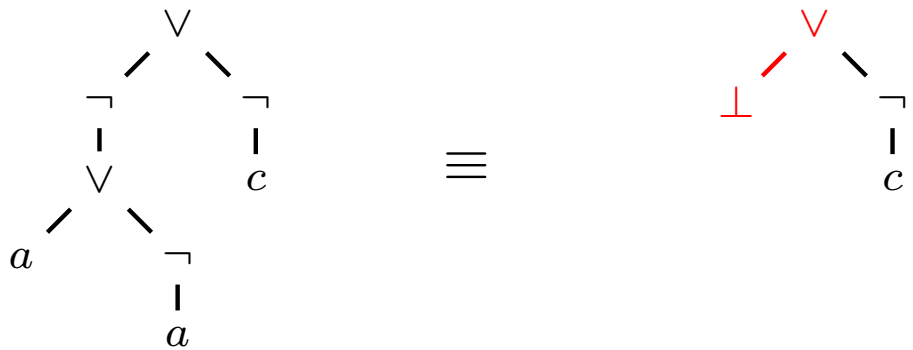

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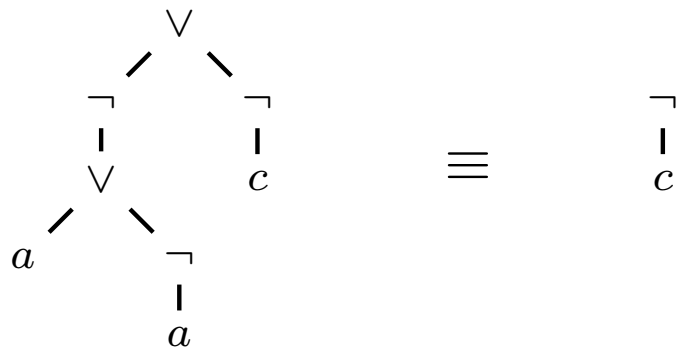

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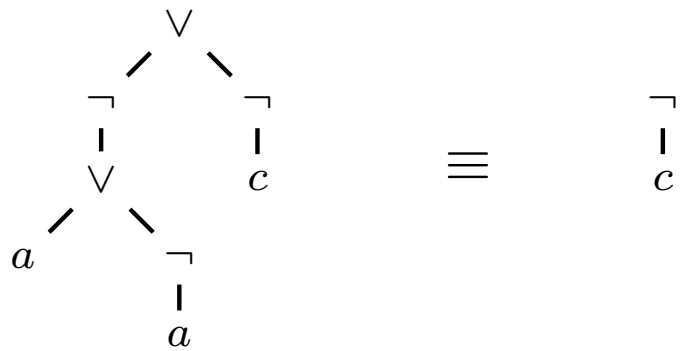

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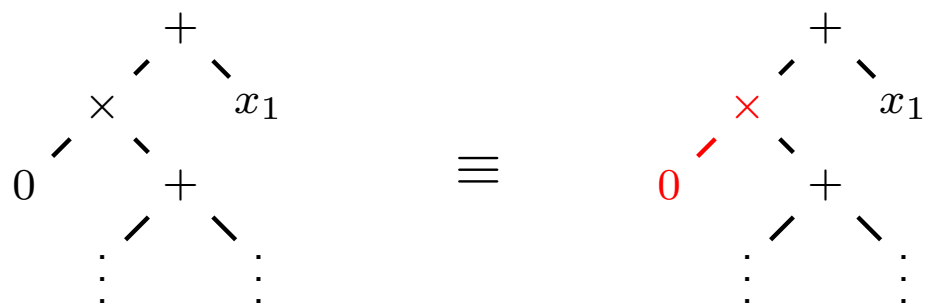

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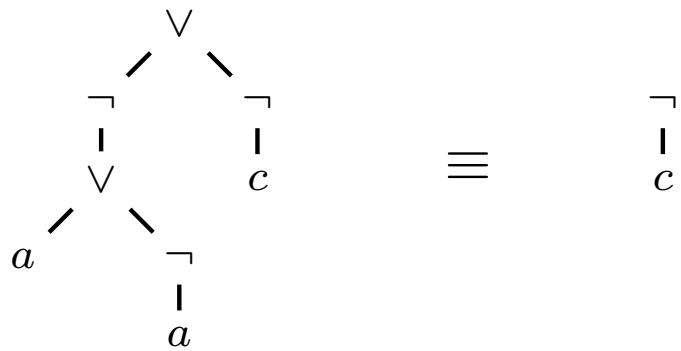

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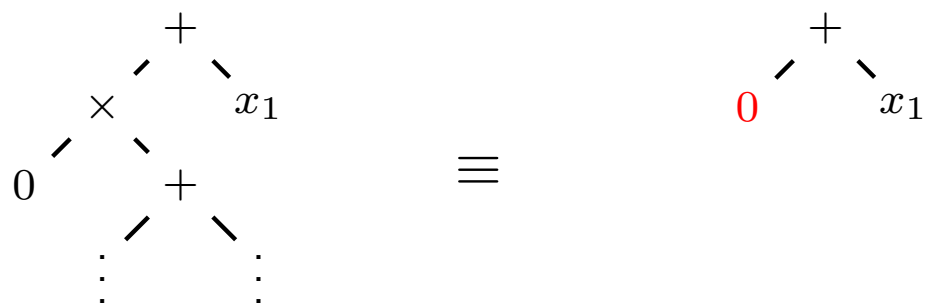

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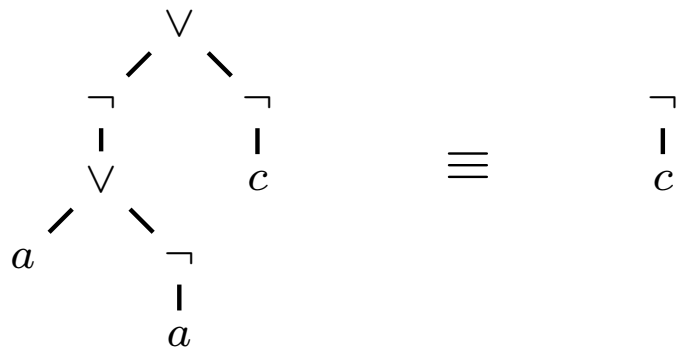

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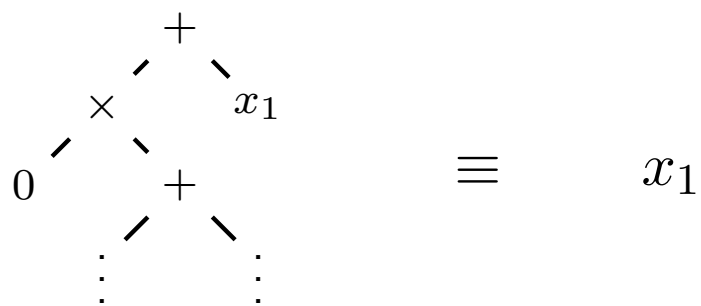

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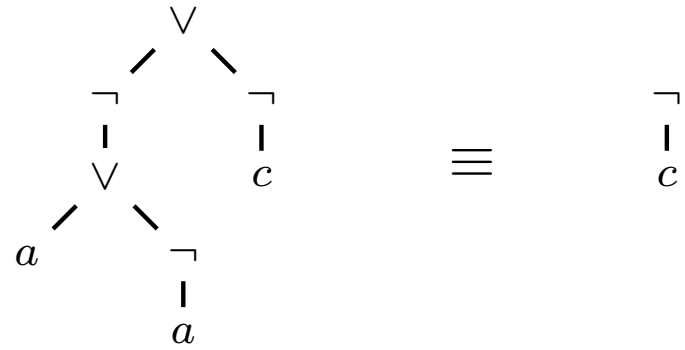

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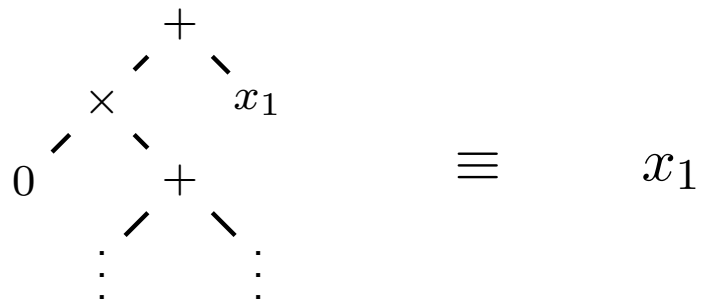

Question

Do semantic reductions affect the size of the random expressions?

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Semantic simplification on uniform trees

Universal result for uniform tree model:

Theorem (Koechlin,Nicaud,R,'20)

Expected size of reduction of uniform tree bounded, as size $\to \infty$.

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$$\overset{\circledast}{\underset{\mathcal{P}}{/}\backslash\underset{T}{}} \rightsquigarrow \mathcal{P} \qquad \overset{\circledast}{\underset{T}{/}\backslash\underset{\mathcal{P}}{}} \rightsquigarrow \mathcal{P}$$

- Works also for systems and different arities,

$$\texttt{Example} : \begin{cases} \mathcal{L}_R = \overset{\neg}{|}_{\mathcal{S}} + \mathcal{S}, \\ \mathcal{S} = a + b + \overset{\vee}{\underset{\mathcal{L}_R}{/}\backslash\underset{\mathcal{L}_R}{}} + \overset{\wedge}{\underset{\mathcal{L}_R}{/}\backslash\underset{\mathcal{L}_R}{}}. \end{cases}$$

Absorbing patters: simplifying the trees

Definition (Simplification, absorbing pattern)

To define an absorbing pattern we fix:

- an "operator" $\circledast \in \mathcal{A}_{ops}$ with arity $(\geq)2$,
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Simplify by applying bottom-up the rule:

$$\begin{array}{ccc} & \circledast & \\ / & & \backslash \\ C_1 & & C_2 \end{array} \rightsquigarrow \mathcal{P}\text{, whenever } C_i = \mathcal{P} \text{ for some } i \in \{1, 2\}.$$

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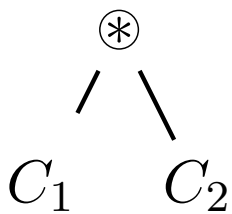
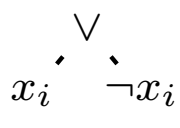
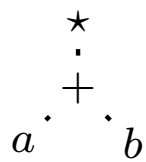


$\rightsquigarrow \mathcal{P}$, whenever $C_i = \mathcal{P}$ for some $i \in \{1, 2\}$.

- Wide variety of examples:



operator $\vee$



operator $+$

$x \mapsto 0$

operator $\times$

- Weak hypothesis: in practice often several coexisting patterns !

Regular Expressions

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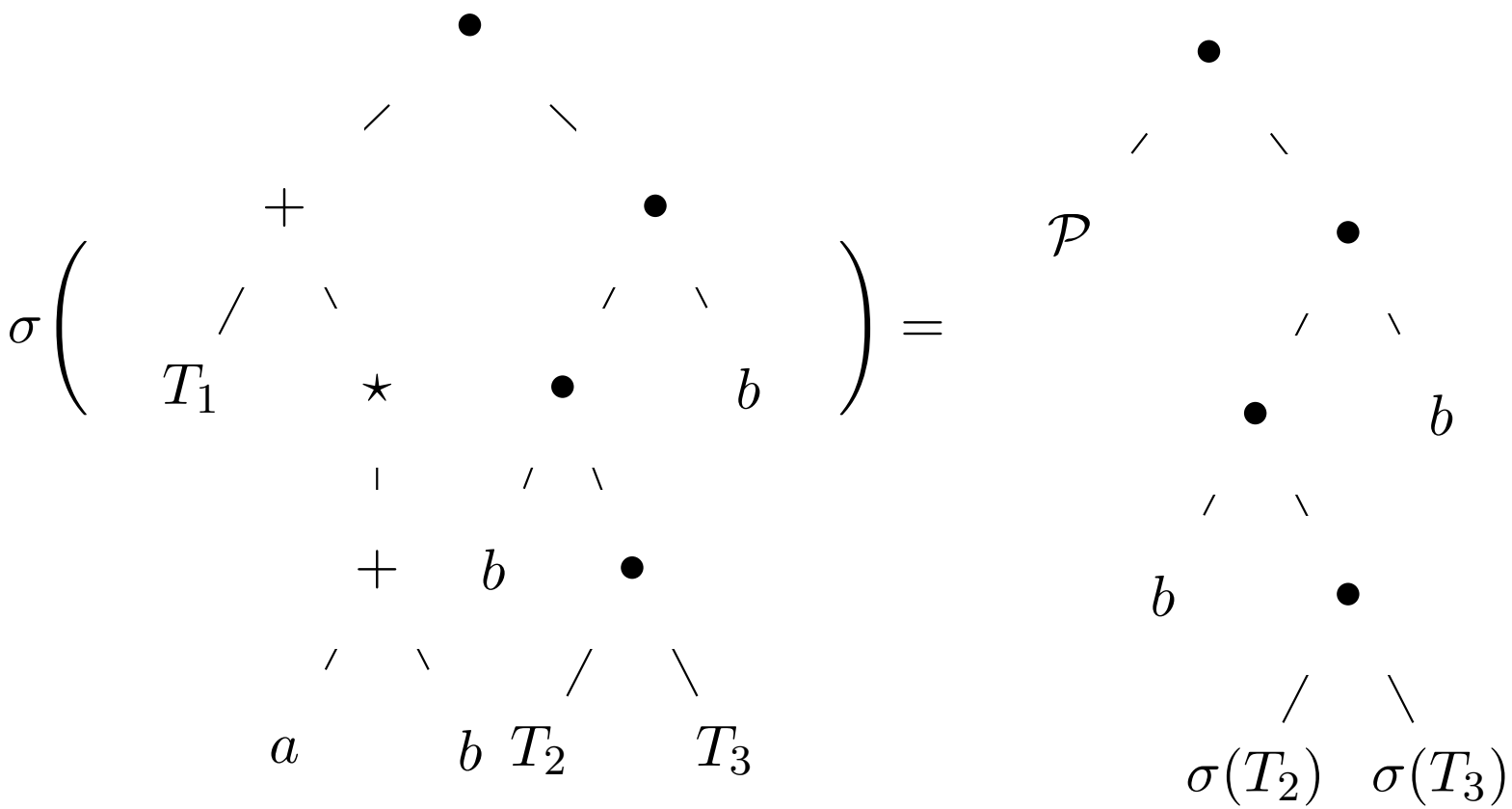
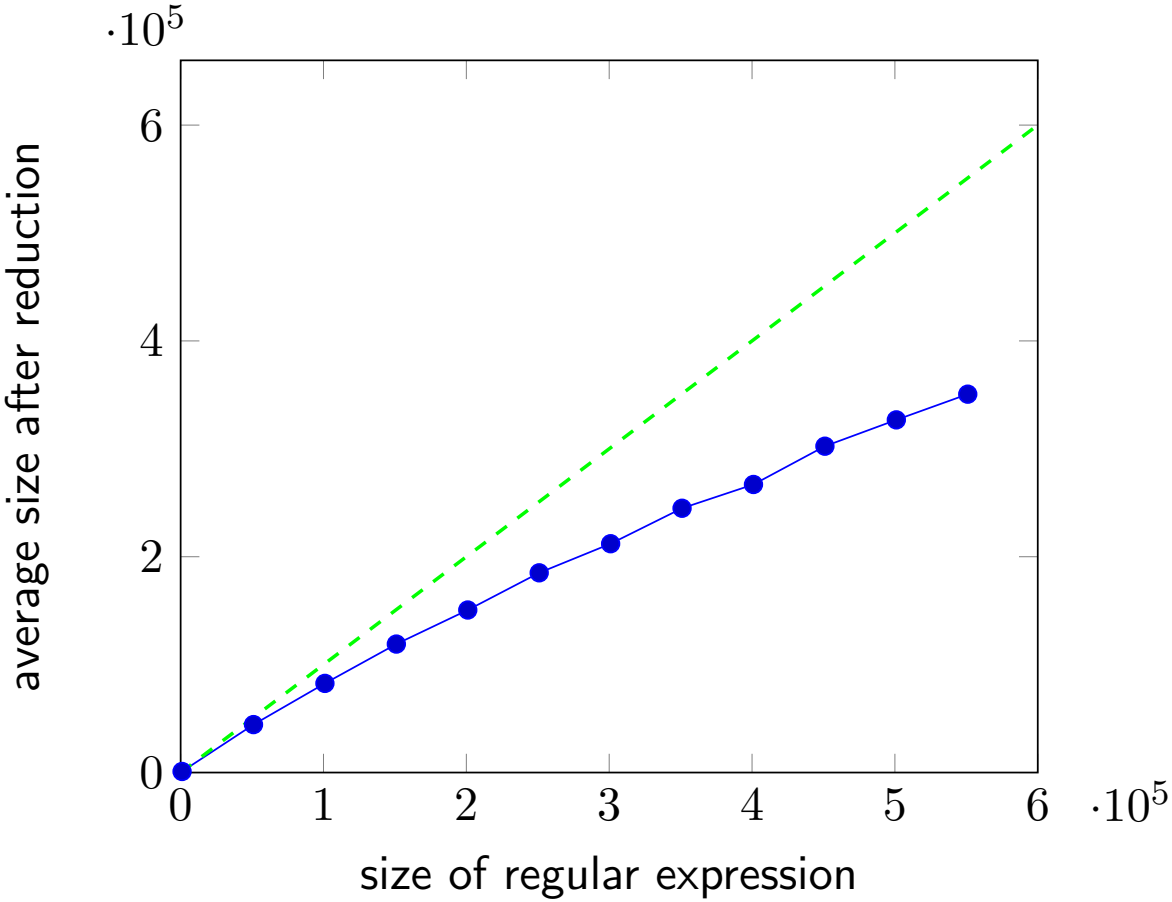
Example: regular expressions $(+, \bullet, \star)$ on two letters a, b:

$\mathcal{P} = (a + b)^\star$ absorbing for union $\circledast = +$



Regular Expressions II

Hidden constant $O(1)$: for the set of all regular expressions tree on two letters, the limit size after reduction is $3\ 624\ 217$.



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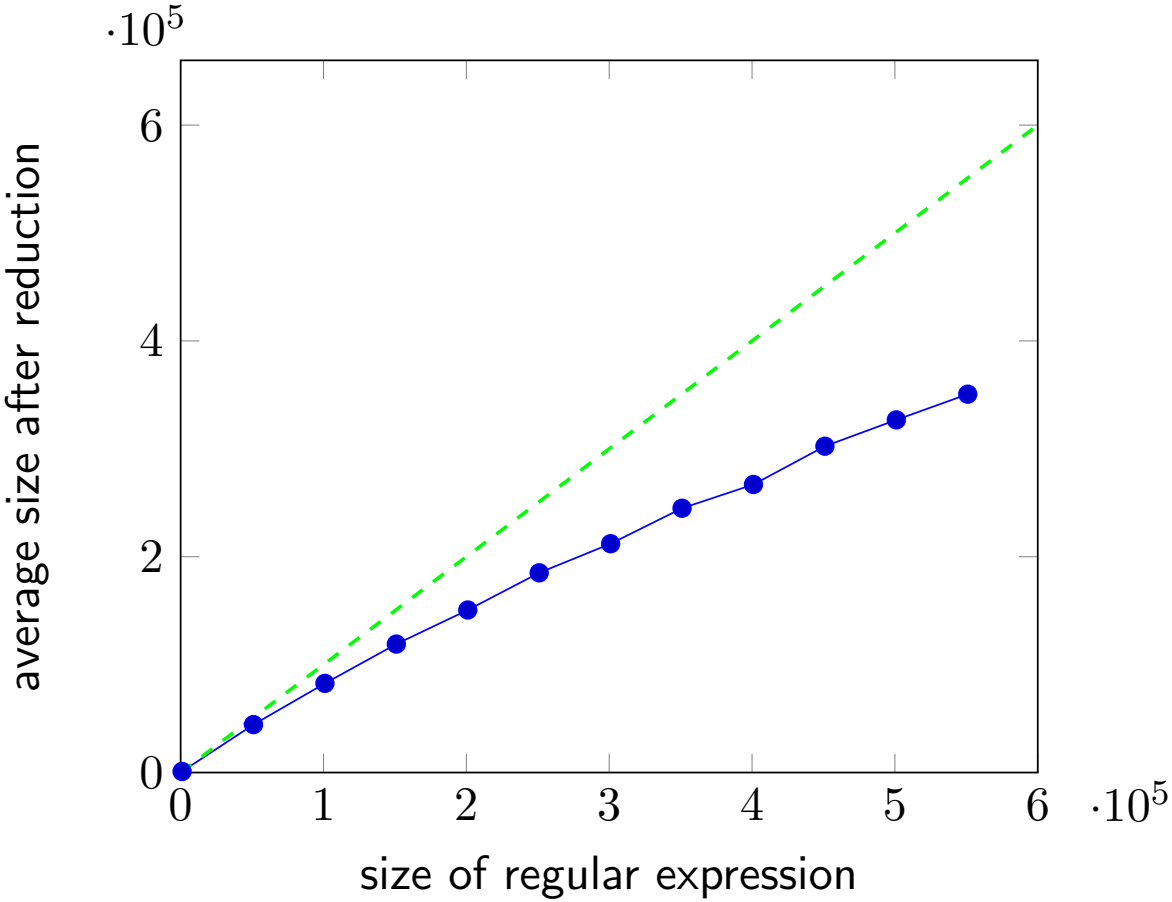

Question. Are uniform regular expressions useful nonetheless?

Regular Expressions III

Extend idea: substitute universal subtrees[2] by tree $\mathcal{P}$ corresponding to $\Sigma^\star$.

[2] The associated language is $\Sigma^\star$, all words.

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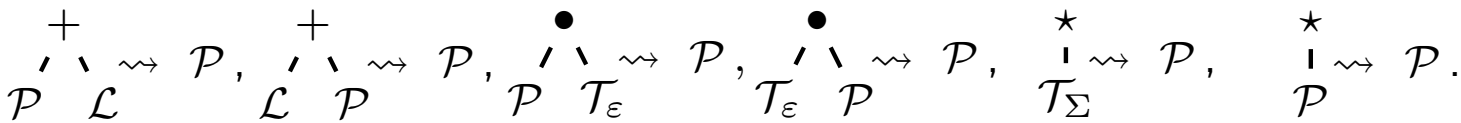

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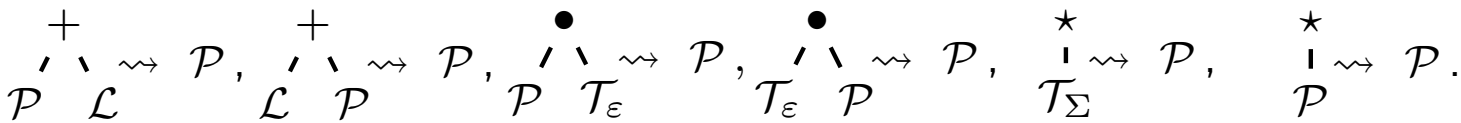

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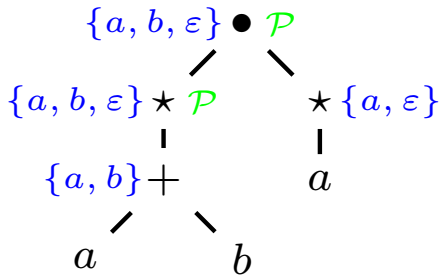
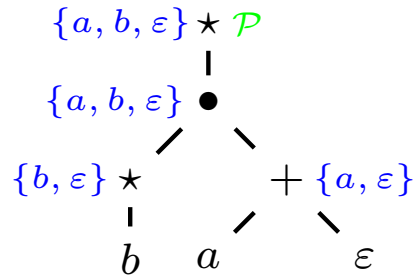


(I) : $(a + b)^\star \cdot a^\star$



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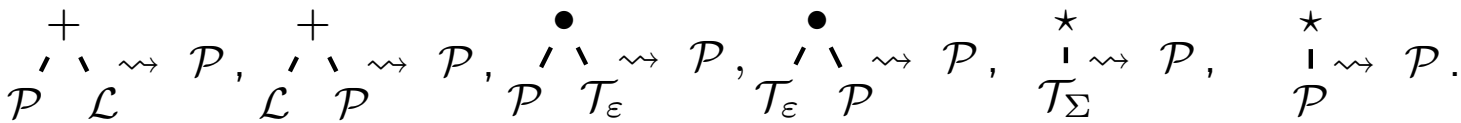

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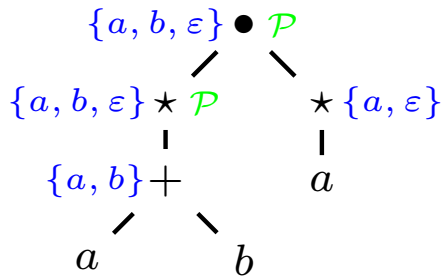

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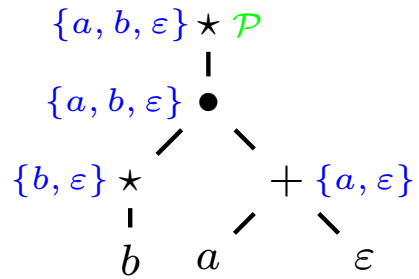

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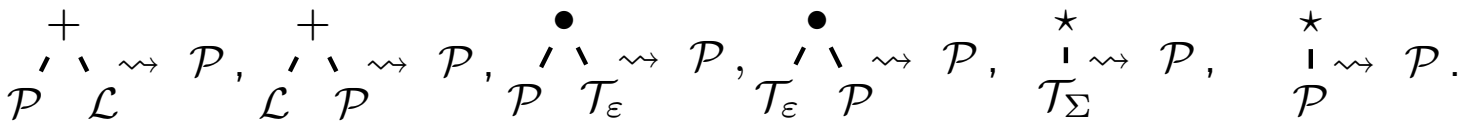

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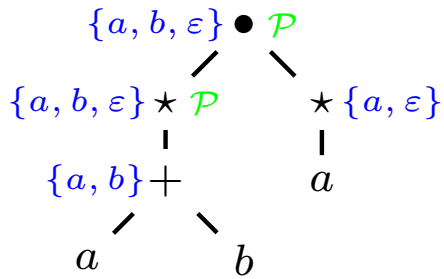

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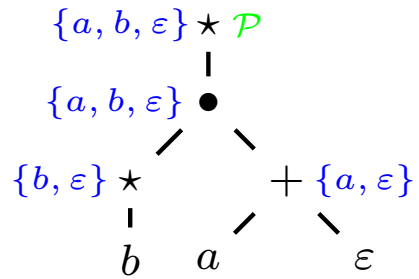

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- Detection only partial: universality problem is PSPACE-complete.
- Yet average size of reduced **unifrom** expression ≈ 77.8 .

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Systems of Regexp [Koechlin, Pivoteau, R. '25]

We compute the probabilities of

- reducing to $\mathcal{P}$: denoted $\mathbb{P}_S^{\langle \mathcal{U} \rangle}$
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Consider the grammar preventing the redundancies from tower of $\star$ and

1. the associativity of $+$,
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$\Rightarrow$ we provide automatic tool to test *overabundance* of universal expressions.

Section

Zoom into the case of BST-like trees

If we draw a random BST-like tree expression of size n:

- do we have the same flaw as uniform trees?

[3] Left to right
$(p_\star, p_\bullet, p_+) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$,$(p_\star, p_\bullet, p_+) = (\frac{5}{29}, \frac{5}{29}, \frac{19}{29})$,$(p_\star, p_\bullet, p_+) = (\frac{1}{10}, \frac{1}{10}, \frac{8}{10})$

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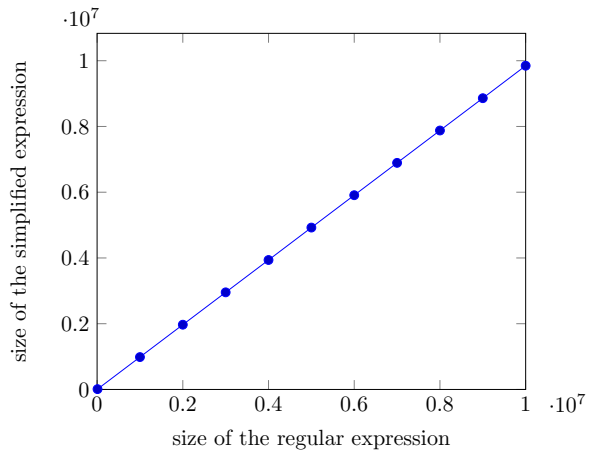
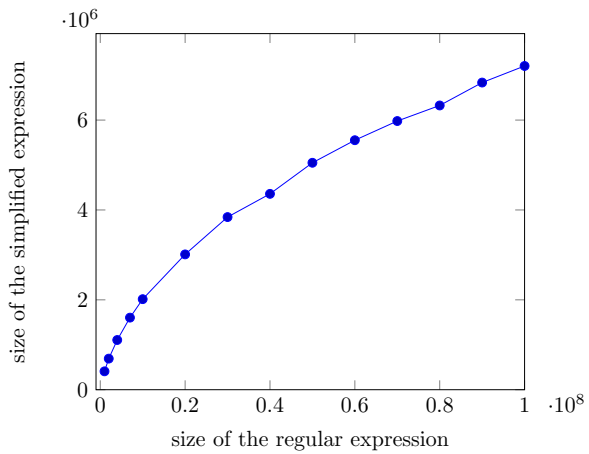
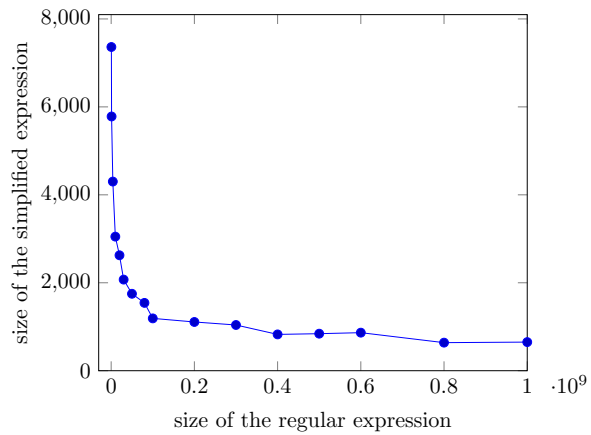
Experimental expected size (10 000 samples)[3] on regular expressions $(+, \bullet, \star)$ on two letters a, b:

$\mathcal{P} = (a + b)^\star$ absorbing for union $\circledast = +$







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Theorem. *Consider a family of expression trees defined from unary and binary operators with an absorbing pattern $\mathcal{P}$ for an operator $\circledast$ of arity 2.*

Take the simplification consisting in inductively changing a $\circledast$-node by $\mathcal{P}$ whenever one of its children simplifies to $\mathcal{P}$.

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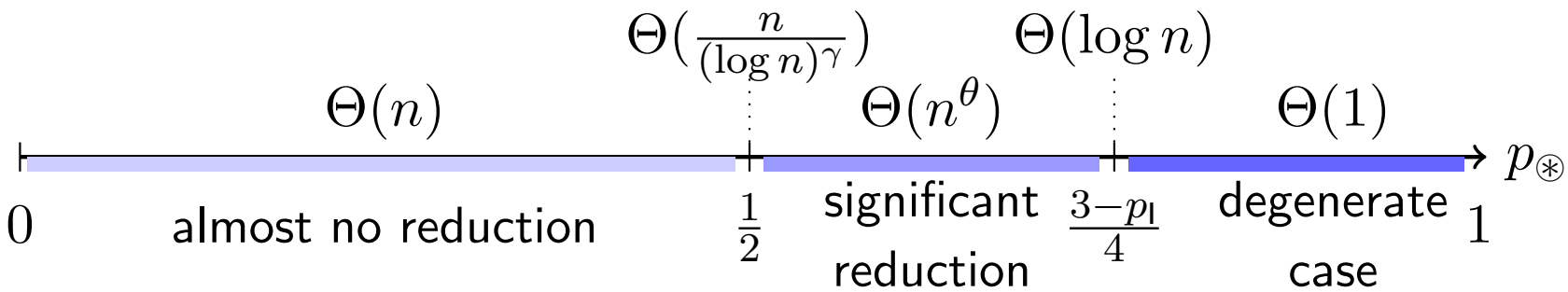

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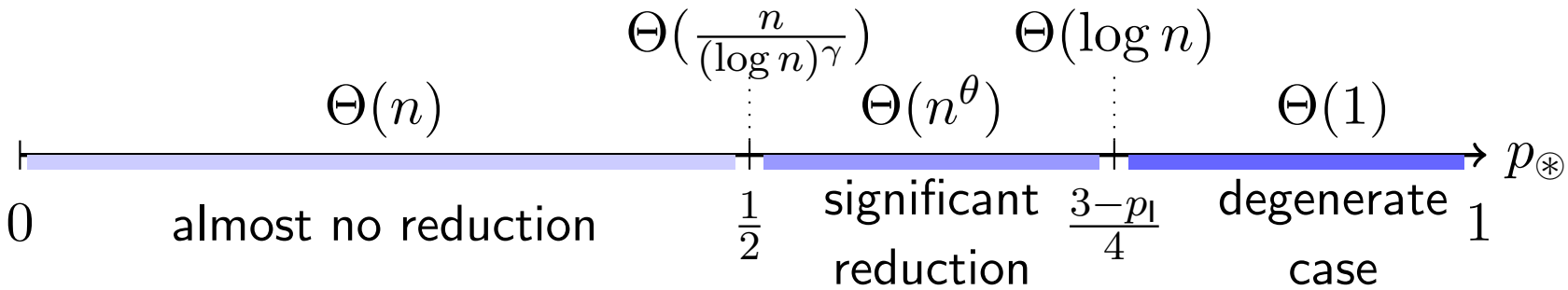
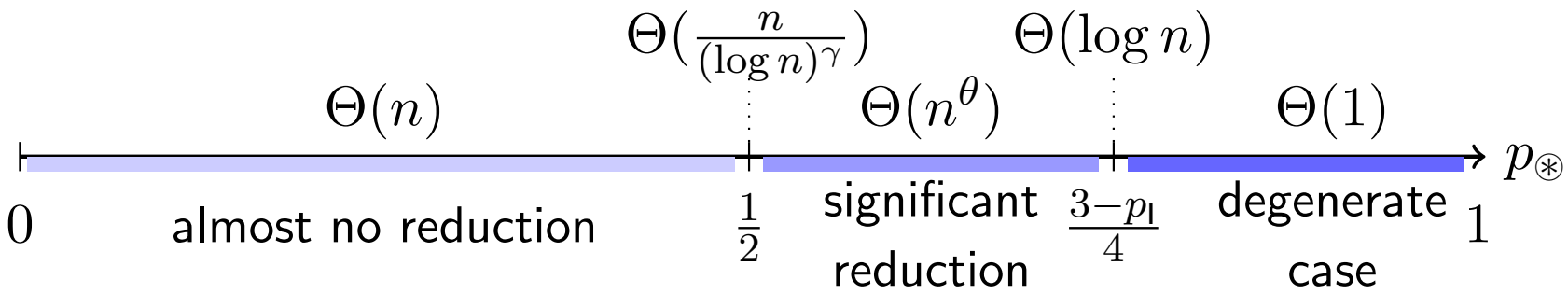


- Probability $p_\circledast$ of $\circledast$, and p_l of picking unary operator.
- Two critical points $p_\circledast = 1/2$ and $p_\circledast = (3 - p_\mathsf{l})/4$
- Regimes from no reduction $\Theta(n)$ to complete reduction $\Theta(1)$

The main regimes experimentally



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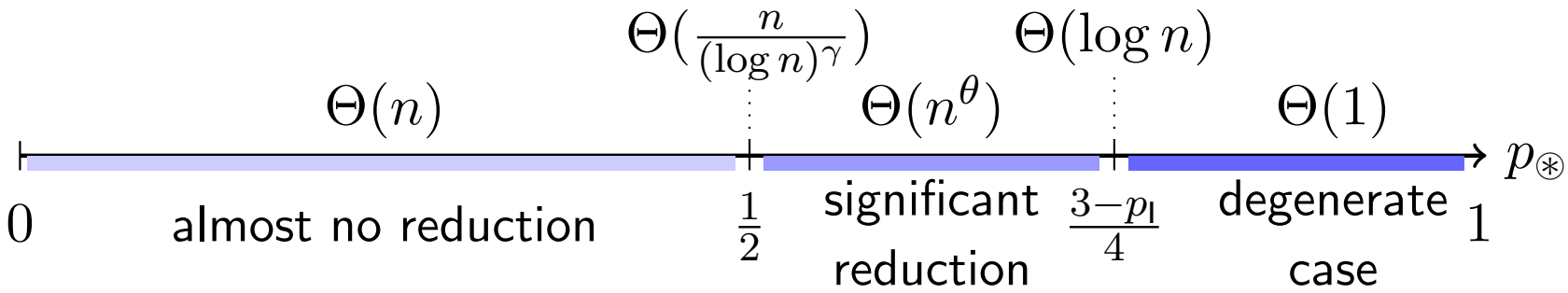
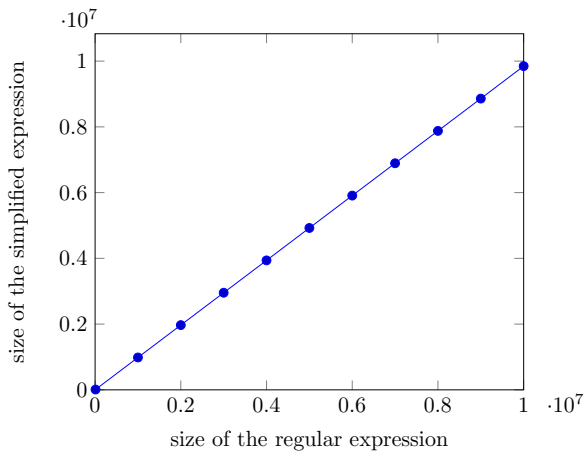
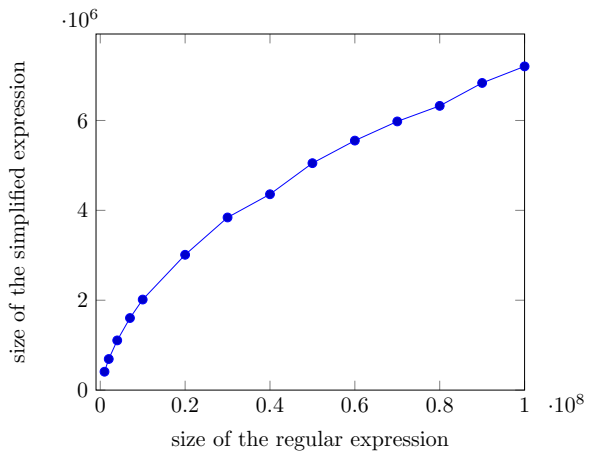


Experimental plots (10 000 samples)for regular expressions on two letters a, b: $\mathcal{P} = (a + b)^\star$ absorbing for union $\circledast = +$





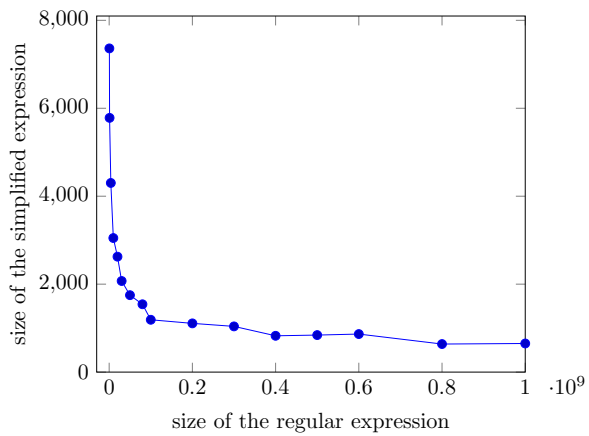


Figure: Left to right: linear ($p_+ = p_\star = p_\cdot = \frac{1}{3}$), sublinear ($p_+ = \frac{19}{29}$, $p_\star = p_\cdot = \frac{5}{29}$) and constant ($p_+ = \frac{8}{10}$, $p_\star = p_\cdot = \frac{1}{10}$).

Scheme of the proof: steps from Analytic combinatorics

We employ Analytic Combinatorics to study the expectation,

- Ordinary generating function

$$E(z) := \sum_{n=0}^{\infty} e_n z^n \,,$$

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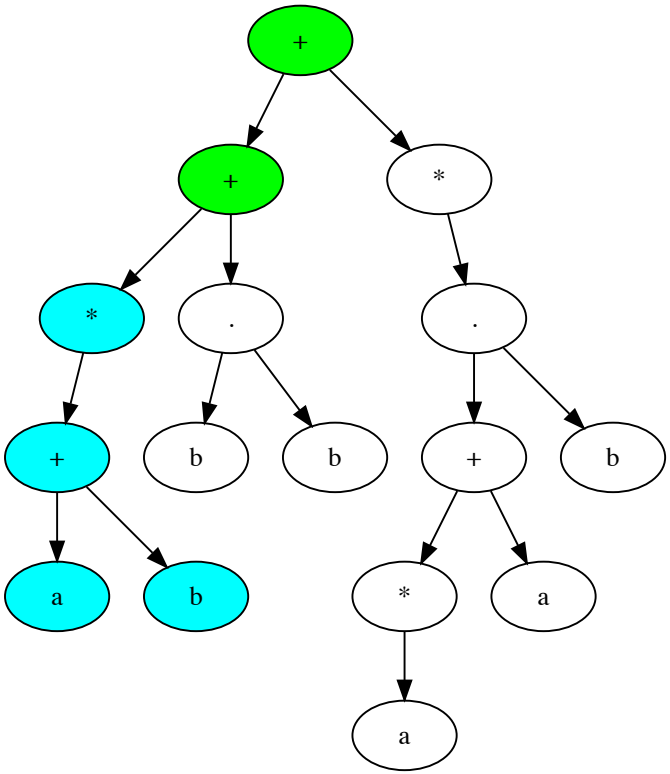
- **Analytic Step.** We look at $E(z)$ over the complex $z \in \mathbb{C}$.

A *Transfer Theorem* links the behaviour of $E(z)$ at its dominant singularity to asymptotics of $e_n \Rightarrow$ Study singularities

$$E(z) \sim_{z\to 1} \lambda(1-z)^{-\alpha} \Longrightarrow e_n \sim \lambda n^{\alpha-1}/\Gamma(\alpha)$$

Symbolic step: fully reducible trees

An expression tree T is *fully reducible* when $\sigma(T) = \mathcal{P}$.



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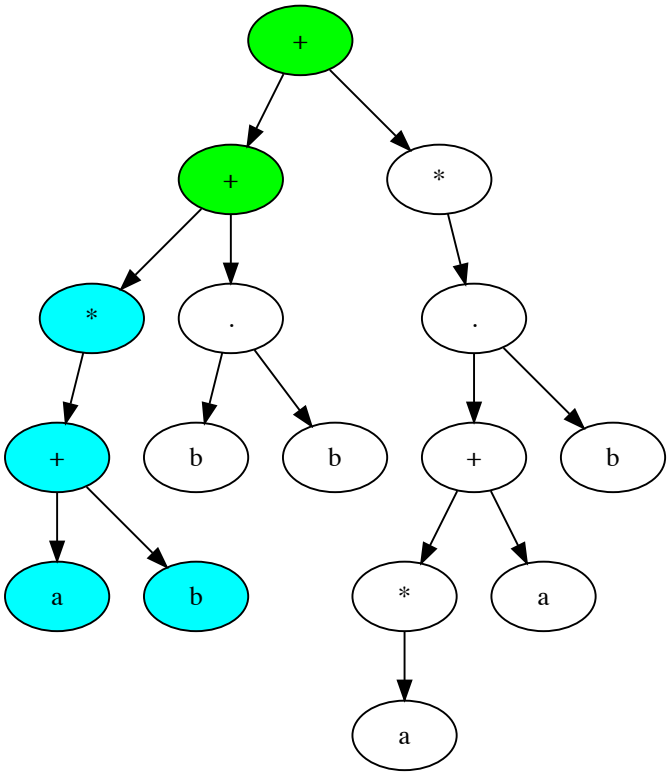

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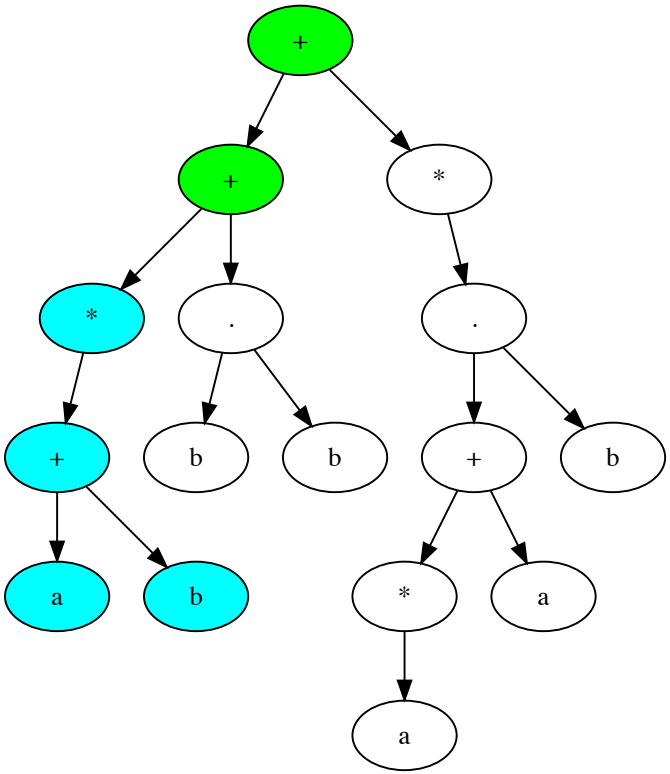

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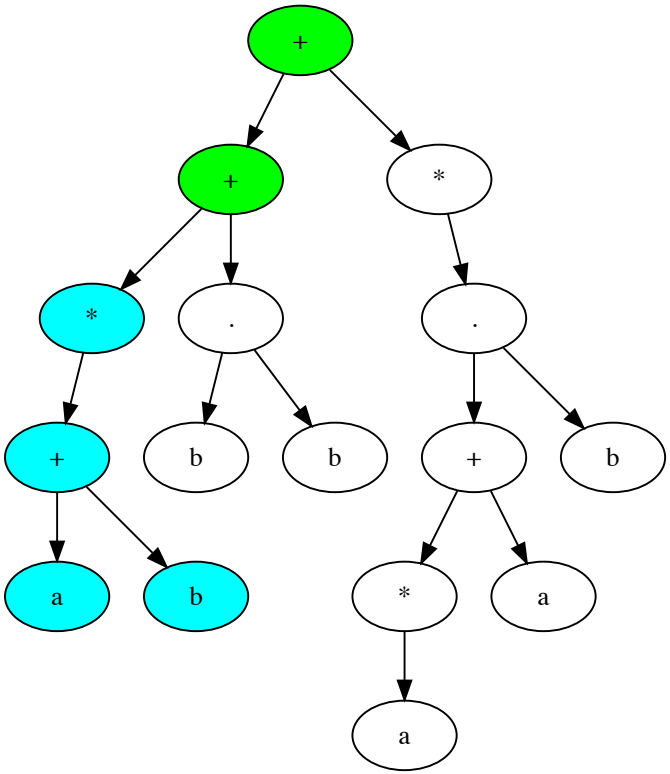

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We consider a fundamental sequence

$$\gamma_n := \Pr_n \{\sigma(T) = \mathcal{P}\} \, , \qquad A(z) := \sum \gamma_n z^n \, ,$$

of probabilities of *full* reduction and their generating function.

Symbolic step: recurrence = differential equation

Recurrence for e_n in terms of γ_n yields first order differential equation

$$E'(z) = F(z, A(z)) + \tfrac{1}{1-p_{\mathsf{I}} z}\left(\tfrac{2}{z} - p_{\mathsf{I}} + 2\,(1-p_{\mathsf{I}})\,\frac{z}{1-z} - 2p_{\circledast} A(z)\right)\cdot E(z)\,,$$

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First order differential equations can be solved explicitly:

Proposition

Let f, g analytic on a star-shaped Ω and consider

$$U'(z) = f(z) + g(z)U(z)\,.$$

The equation has a unique solution analytic on Ω, with $U(0) = u_0$,

$$U(z) = \exp\left(\int_0^z g(\zeta)d\zeta\right)\left(u_0 + \int_0^z f(\zeta)\exp\left(-\int_0^\zeta g(w)dw\right)d\zeta\right).$$

Symbolic step: recurrence = differential equation

Recurrence for e_n in terms of γ_n yields first order differential equation

$$E'(z) = F(z, A(z)) + \tfrac{1}{1-p_{\mathsf{I}} z}\left(\tfrac{2}{z} - p_{\mathsf{I}} + 2\,(1-p_{\mathsf{I}})\,\frac{z}{1-z} - 2p_{\circledast} A(z)\right)\cdot E(z)\,,$$

in terms of $A(z) = \sum_n \gamma_n z^n$.

First order differential equations can be solved explicitly:

Proposition

Let f, g analytic on a star-shaped Ω and consider

$$U'(z) = f(z) + g(z)U(z)\,.$$

The equation has a unique solution analytic on Ω, with $U(0) = u_0$,

$$U(z) = \exp\left(\int_0^z g(\zeta)d\zeta\right)\left(u_0 + \int_0^z f(\zeta)\exp\left(-\int_0^\zeta g(w)dw\right)d\zeta\right).$$

Our coefficients depend on z and the unknown generating function $A(z)$.

Analytic step: the singularity $z = 1$

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Solution of ODE yields, as $z \to 1$

$$E(z) \sim \frac{c}{(1-z)^2} \left(2 + \int_0^z F(w, A(w)) I(w) dw \right) (I(z))^{-1} , ,$$

where $I(z) := \exp\left(2p_{\circledast} \int_0^z \frac{A(w)}{1-p_{\mathsf{l}} w} dw \right)$.

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To apply the Transfer Theorem and complete the proof:

- we require precise asymptotics for $A(z)$ at $z = 1$,
- we show that $A(z)$ and $E(z)$ are analytic over $\Omega = \mathbb{C} \setminus [1, \infty)$.

Fully reducible trees: probabilities

We study the generating function $A(z) = \sum \gamma_n z^n$

Proposition: recurrence for γ_n

The probabilities $\gamma_n = \Pr_n \{\sigma(T) = \mathcal{P}\}$ satisfy, for $n \geq |\mathcal{P}|$,

$$\gamma_{n+1} = \frac{p_{\circledast}}{n-1} \sum_{k=1}^{n-1} (\gamma_k + \gamma_{n-k} - \gamma_k \gamma_{n-k}) \,.$$

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Recurrence translates into Riccati differential equation

$$A'(z) = (s-2)\gamma_s z^{s-1} + \left(\frac{2}{z} + 2p_\circledast \frac{z}{1-z} \right) A(z) - p_\circledast \cdot (A(z))^2 \,,$$

where $s = |\mathcal{P}|$ is the size of the absorbing pattern.

Behaviour of solutions of Riccati ODE

Considering $v(z)$ such that $p_{\circledast} A(z) = v'(z)/v(z)$,

Riccati equation becomes linear

$$v''(z) = p_{\circledast} \cdot (s-2)\gamma_s z^{s-1} v(z) + \left(\frac{2}{z} + 2p_{\circledast} \frac{z}{1-z} \right) v'(z) \,.$$

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- domain of analyticity well-understood.
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Proposition

The generating function $A(z)$ satisfies, $z \to 1$

- For $p_{\circledast} > \frac{1}{2}$, $A(z) = \frac{\gamma_\infty}{1-z} + O((1-z)^{2p_{\circledast}-2})$,
- For $p_{\circledast} = \frac{1}{2}$, $A(z) = \frac{2}{1-z} \left(\log\left(\frac{1}{1-z}\right) \right)^{-1} \left(1 + O\left(\log\left(\frac{1}{1-z}\right)^{-1} \right) \right)$
- For $p_{\circledast} < \frac{1}{2}$, $A(z) \sim \frac{D}{(1-z)^{2p_{\circledast}}}$,

where $\gamma_\infty := (2p_{\circledast} - 1)/p_{\circledast}$ and $D > 0$ is a constant.

Probability of full reduction

Theorem

The probability γ_n of being fully reducible tends to the constant $\gamma_\infty := (2p_\circledast - 1)/p_\circledast$ for $p_\circledast > \frac{1}{2}$ and to zero if $p_\circledast \leq \frac{1}{2}$.

Moreover, for the cases in which it tends to zero,

- *for $p_\circledast = \frac{1}{2}$ we have $\gamma_n \sim \frac{2}{\log n}$,*
- *for $p_\circledast < \frac{1}{2}$, $\gamma_n \sim D \cdot n^{2p_\circledast - 1}$.*

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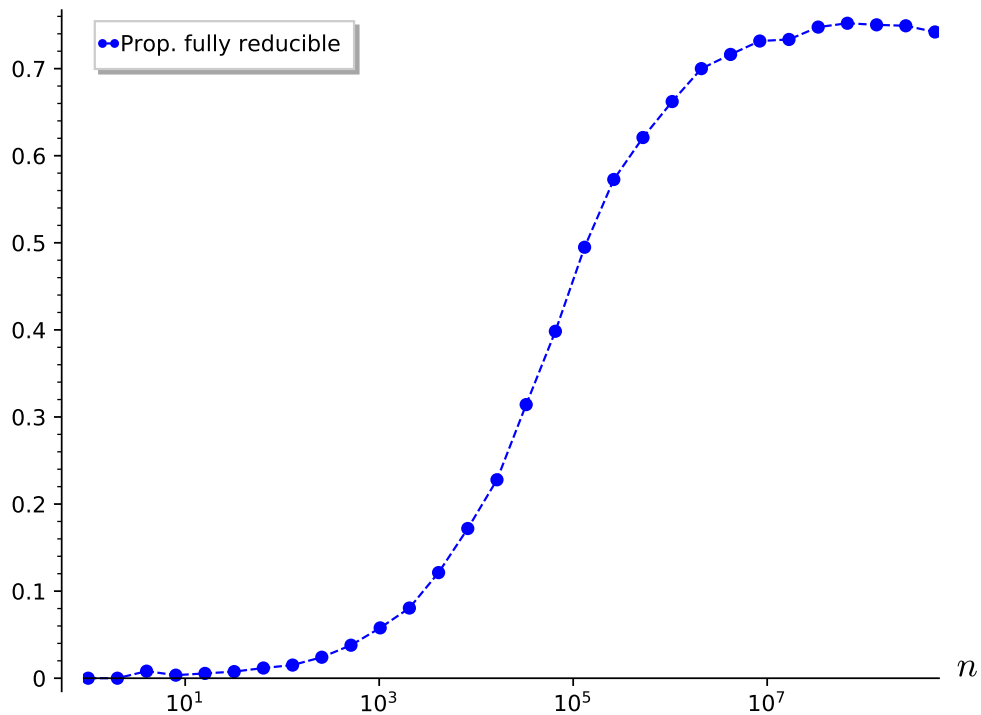
Experimental plot:
regular expressions on two letters with

$$(p_+, p_\bullet, p_\star) = (\tfrac{8}{10}, \tfrac{1}{10}, \tfrac{1}{10})\,.$$

Then

$$\lim_{n\to\infty} \gamma_n = 3/4\,.$$



Section

Conclusion and further work

- A simple linear algorithm, reduces uniform regular expressions to small constant size.
- Uniform random regular expression trees tend to describe very limited languages.
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   $\Rightarrow$ consider interactions between several operators?
3. Take a concrete case: LTL formulas.

References

Florent Koechlin, Cyril Nicaud, and Pablo Rotondo.
On the Degeneracy of Random Expressions Specified by Systems of Combinatorial Equations.
Proceedings of DLT 2020, LNCS vol. 12086, pp 164–177.
https://doi.org/10.1007/978-3-030-48516-0_13

Florent Koechlin and Pablo Rotondo.
Absorbing patterns in BST-like expression-trees.
Proceedings of STACS 2021, LIPICs, vol.187, 48:1–48:15
https://doi.org/10.4230/LIPIcs.STACS.2021.48

Florent Koechlin, Carine Pivoteau and Pablo Rotondo.
Heuristic universality detection over regular expressions specified by systems.
Proceedings of DLT 2025, August 2025, pp. 278 – 293.
https://doi.org/10.1007/978-3-032-01475-7_19

Thank you!