Analysis of an efficient reduction algorithm for random regular expressions based on universality detection

Pablo Rotondo

LIGM, Université Gustave Eiffel

Joint work with

Florent Koechlin

CSR 2021,

30 June, 2021.

Plan of the talk

1. Introduction: regular expression trees, uniform distribution
2. Semantic reductions: absorbing patterns, universality
3. Main results: expected size, proportion of universals
4. Techniques for the proof
5. Conclusions and further work

Introduction: context

Problem

Automatically test a program taking regular expressions as input

$$(a+b)\cdot b^\star\,, \qquad (b\cdot(a+\varepsilon))^\star\,, \qquad (a\cdot a^\star)+(b+a)^\star\,.$$

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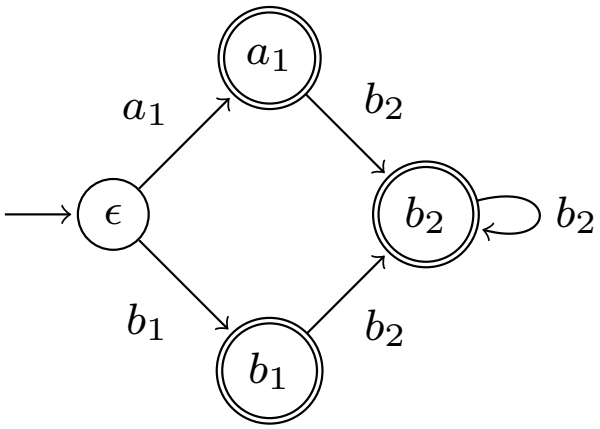
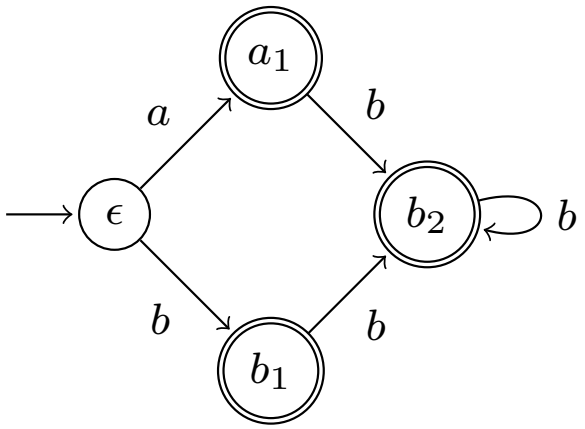
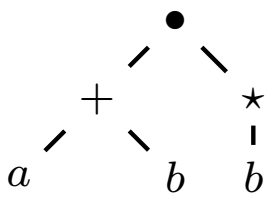
Example: automata constructions





Introduction: random regular expressions

- **Expression trees**



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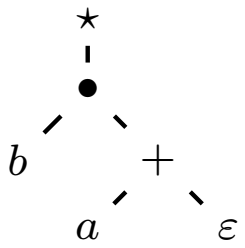

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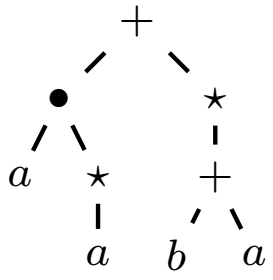

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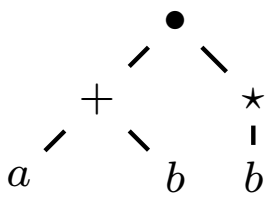

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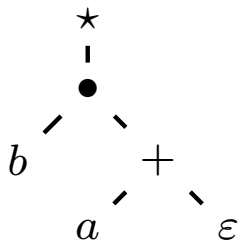

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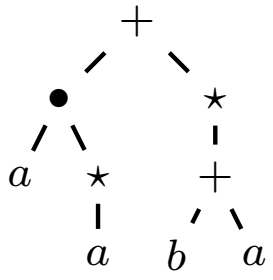

$(a\cdot a^\star)+(b+a)^\star$

- **Generate a random expression tree**
  - Realistic distribution
  - Simple implementation, possibility of theoretical analysis.

Uniform random expression trees

Expression trees:

- trees defined inductively,

$$\mathcal{L} = a_1 + \ldots + a_k + \varepsilon + \underset{\mathcal{L}}{\overset{\star}{|}} + \underset{\mathcal{L}\ \mathcal{L}}{\overset{\bullet}{\wedge}} + \underset{\mathcal{L}\ \mathcal{L}}{\overset{+}{\wedge}},$$

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$\Longrightarrow$ Model used in numerous practical and theoretical works

Warning signs: reduction by an absorbing pattern

Uniform expression trees [Koechlin,Nicaud,R. 2020]

Expected size after (linear) reduction is bounded $O(1)$.

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simplest case, $\texttt{false} \wedge (\ldots) \equiv \texttt{false}$.

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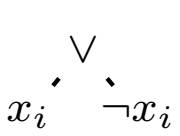
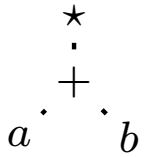


operator $\vee$



operator $+$

$x \mapsto 0$

operator $\times$

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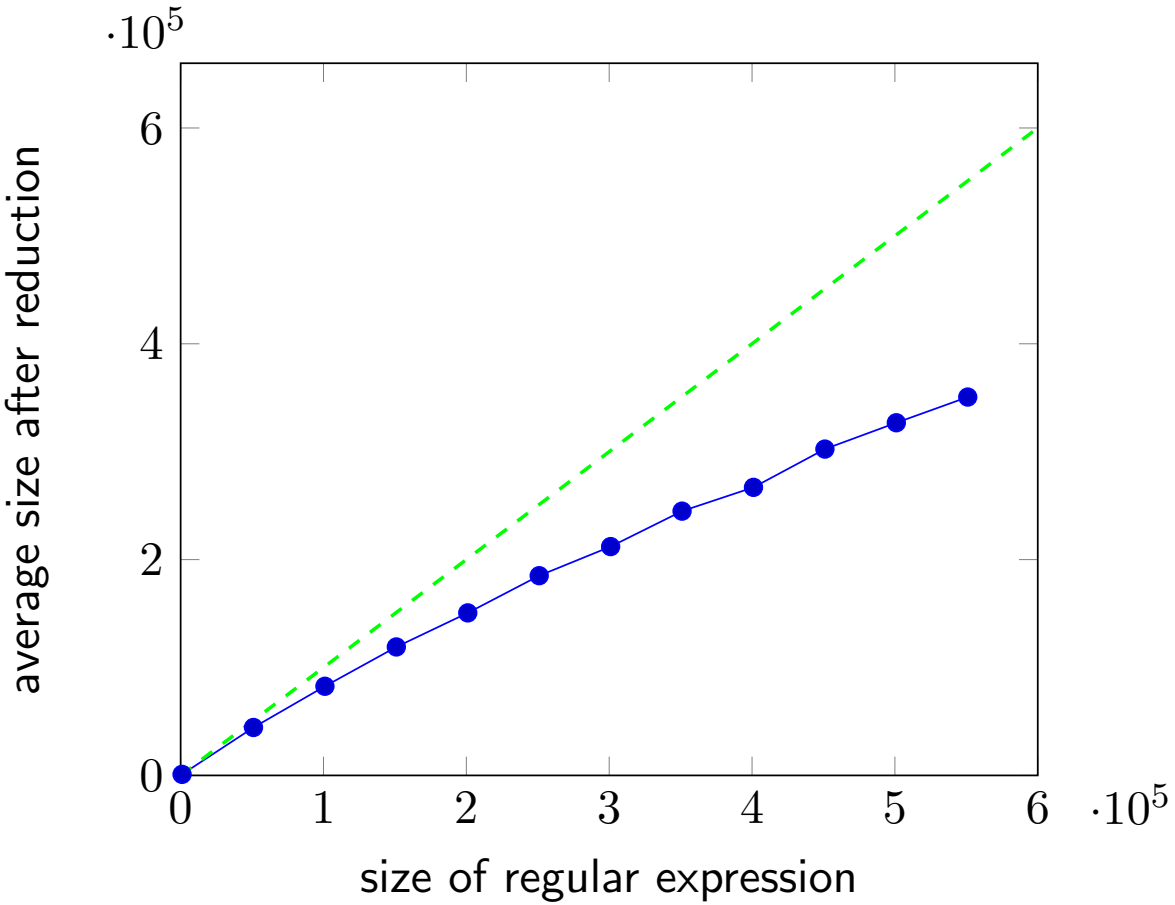
operator $\vee$ | operator $+$ | operator $\times$

What does this say about regular expressions? $O(1)$?

Regular expressions: reduction by absorbing pattern

Hidden constant $O(1)$: for regular expressions on two letters, the limit size after reduction is $3\ 624\ 217$.



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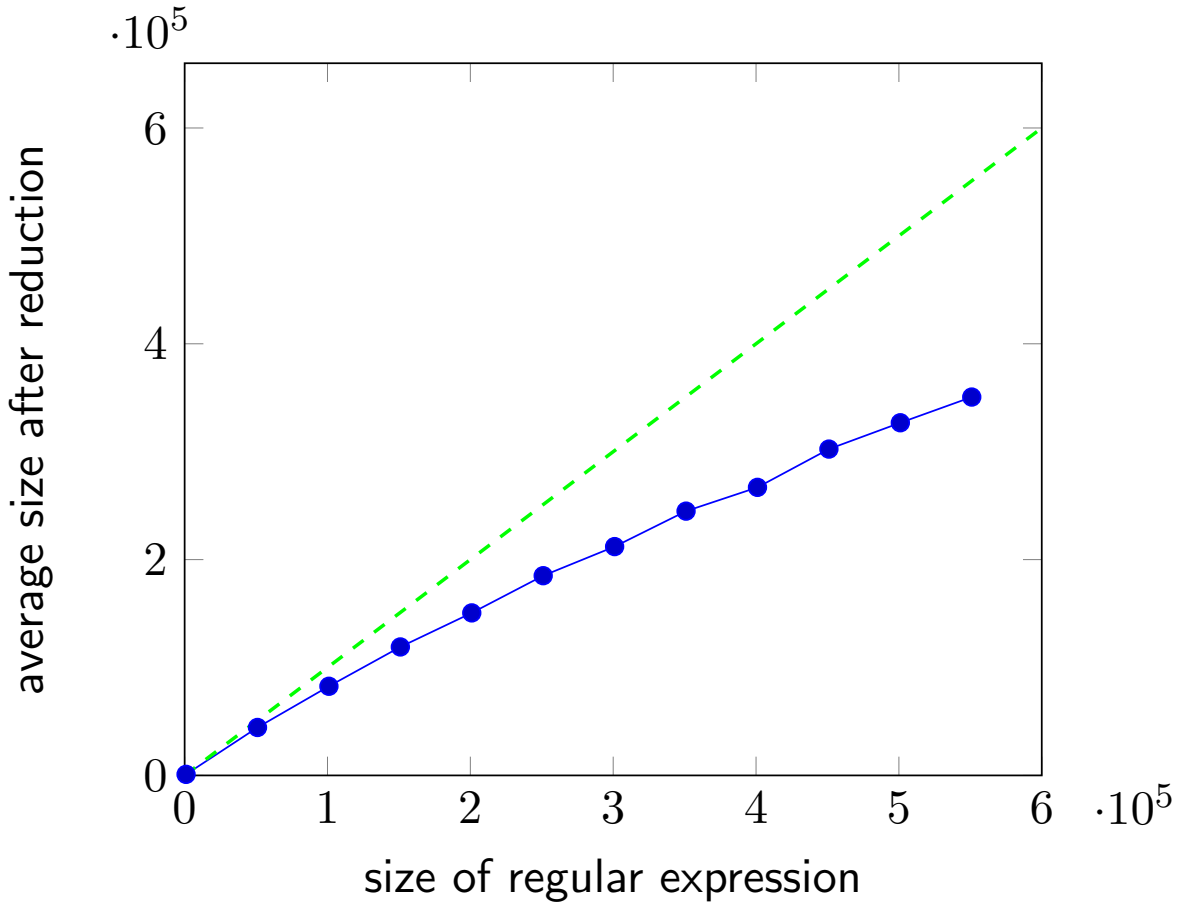
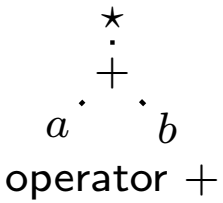


Question. Are uniform regular expressions useful nonetheless?

Reduction based on universality detection

- Reduction from absorbing pattern 

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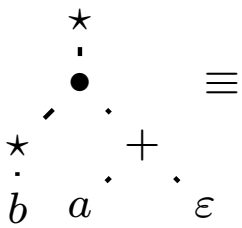
Example: just avoid the pattern



$\equiv \quad \{a, b\}^\star$.

$(b^\star \cdot (a + \varepsilon))^\star$

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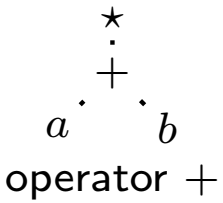
- Reduction from absorbing pattern 

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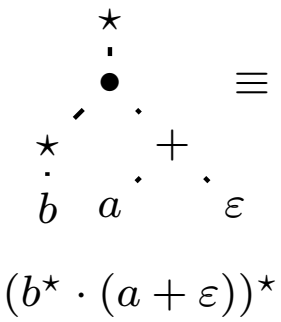

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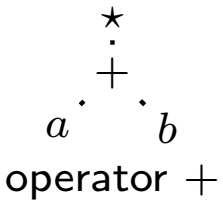
- Reduction from absorbing pattern 

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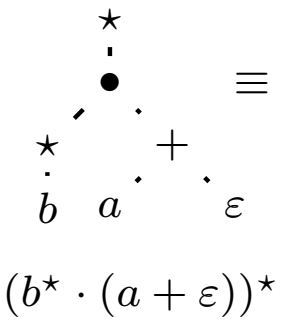

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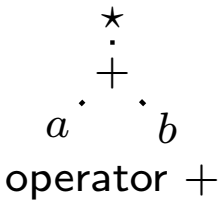
- Reduction from absorbing pattern 

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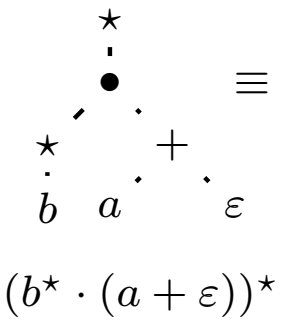

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$\Rightarrow$ substitute universal subtrees by smallest universal tree $\mathcal{U}$.

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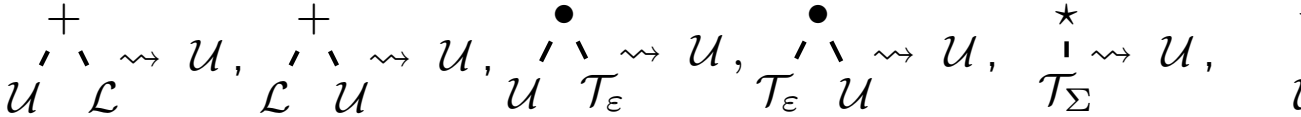

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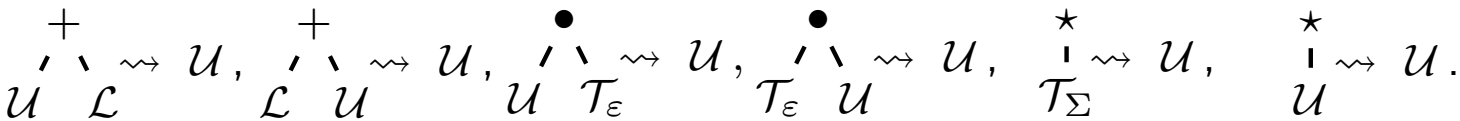

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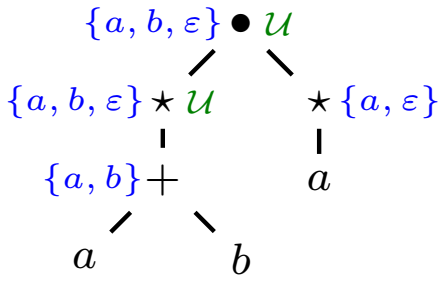

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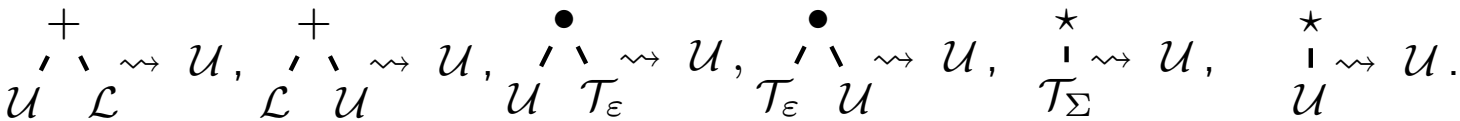

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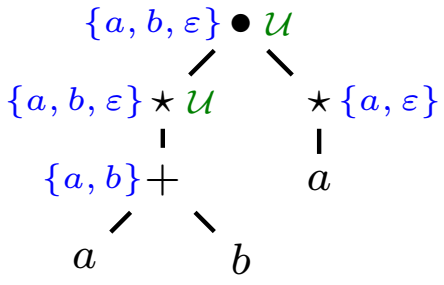

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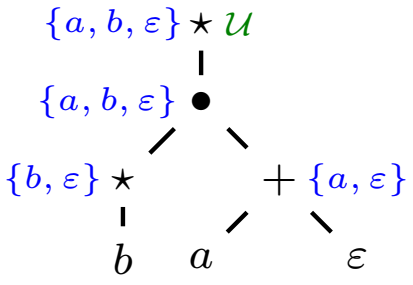

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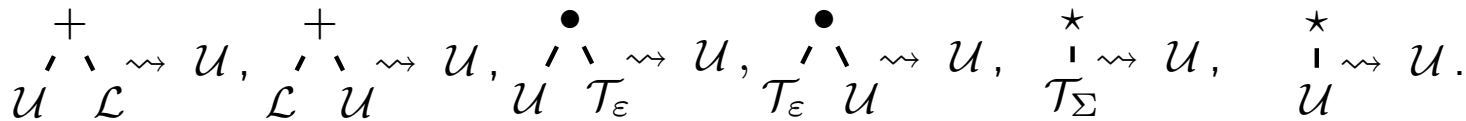

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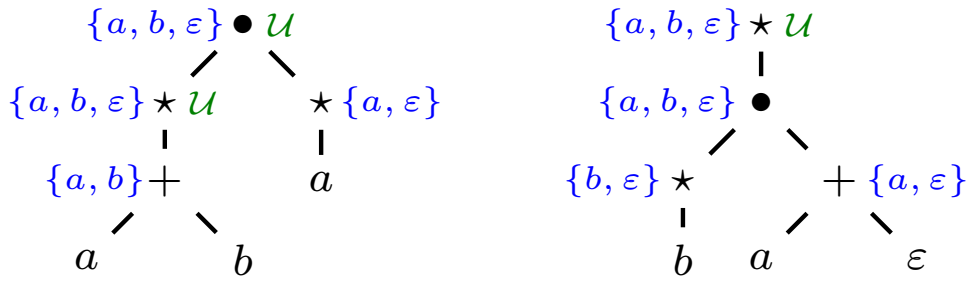

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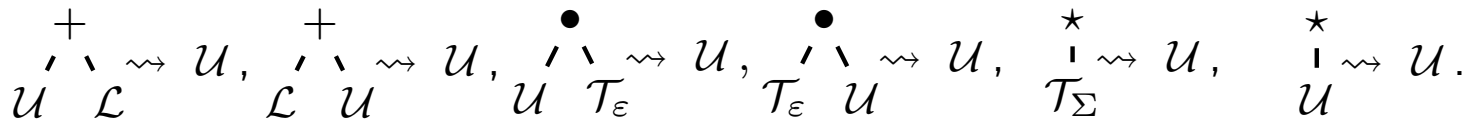

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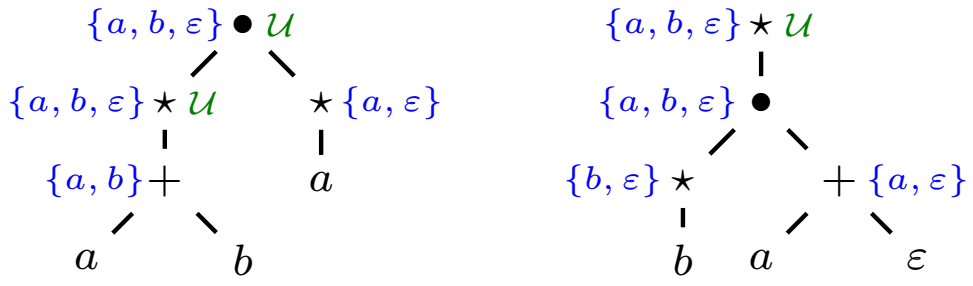

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$\Longrightarrow$ universality problem is PSPACE-complete !

Results I

Rewriting rules:

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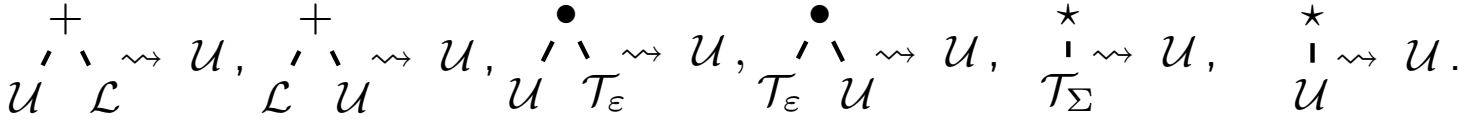

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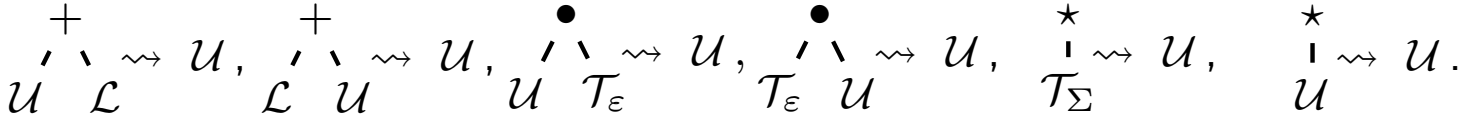

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Then the expected size of the simplification of a random uniform tree tends to a **constant** as the size n tends to infinity.

Moreover, the constant can be computed efficiently

$\lvert\Sigma\rvert$	2	3	4	5
$\lim \mathbb{E}_n[\lvert\sigma(T)\rvert]$	$77.79724\ldots$	$495.59151\ldots$	$2\,518.20513\ldots$	$11\,694.43727\ldots$

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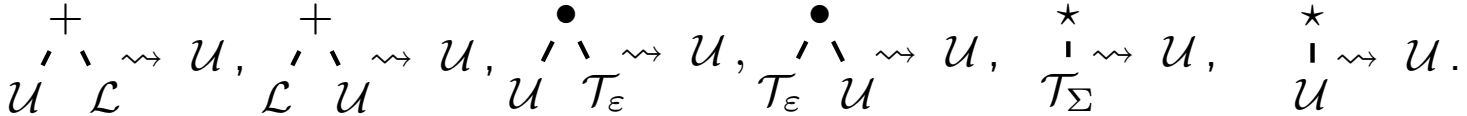

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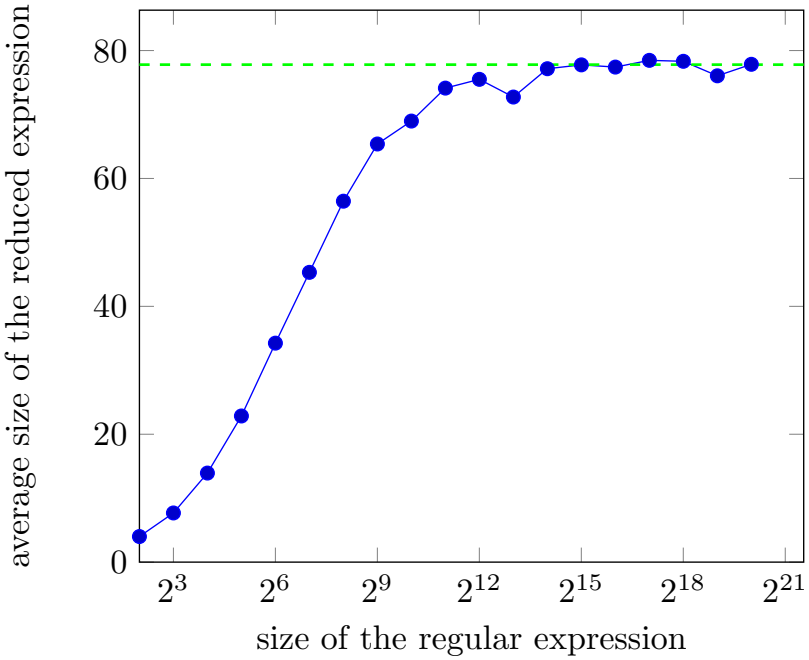
Note. Compare ~ 77.8 against previous $\sim 3.6 \times 10^6$ for two letters.

Results: plots



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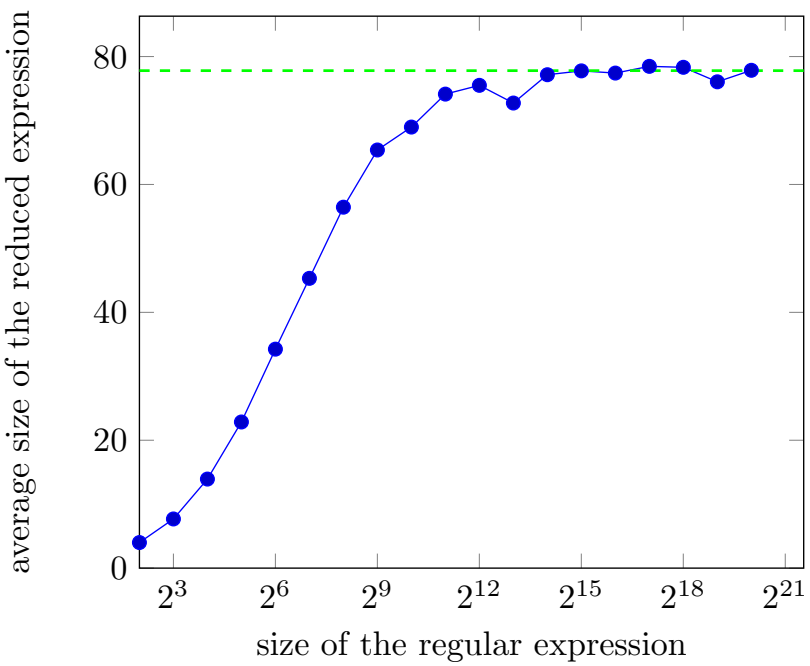

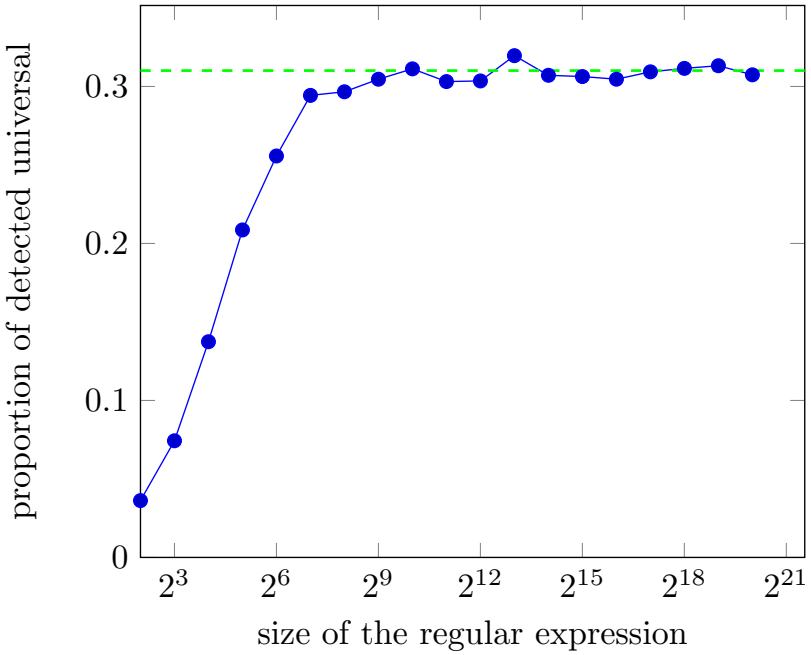

Results II

Proposition

For n large enough, the proportion $\mathrm{Pr}_n(\texttt{univ.})$ of universal expressions trees belongs to the intervals:

$\lvert\Sigma\rvert$	2	3	4	5
`interval`	$(0.31, 0.46)$	$(0.13, 0.27)$	$(0.062, 0.15)$	$(0.028, 0.077)$

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- Preponderance of universal expression trees:
  between 31% and 46% for two letters $\{a, b\}$
- Uniform model not adapted to sampling regular languages

Scheme of the proof

We employ Analytic Combinatorics to study the expectation,

- Bivariate generating function

$$L(z,u) := \sum_{T \in \mathcal{L}} u^{|\sigma(T)|} z^{|T|} \implies \mathbb{E}_n[|\sigma(T)|] = \frac{[z^n]\partial_u L(z,u)|_{u=1}}{[z^n]L(z,u)|_{u=1}} ,$$

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  $\Rightarrow$ Study asymptotics over $z \in \mathbb{C}$

Combinatorial specification: two letters $\Sigma = \{a, b\}$

For every $X \subseteq \{a, b\}$ introduce:

- $\mathcal{T}_{X,\varepsilon}$ the set of trees recognizing every letter in X and ε, and no letter not in X
- $\mathcal{T}_{X,\overline{\varepsilon}}$ the set of trees recognizing every letter in X, and no letter not in X, nor ε

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$$\mathcal{T}_{X,\varepsilon} = \varepsilon \mathbf{1}_{X=\emptyset} + \overset{\star}{\underset{\mathcal{T}_{X,\varepsilon}}{|}} + \overset{\star}{\underset{\mathcal{T}_{X,\overline{\varepsilon}}}{|}} + \sum_{(S,S'):S\cup S'=X} \mathcal{T}_{S,\varepsilon} \overset{\bullet}{\wedge} \mathcal{T}_{S',\varepsilon}$$

$$+ \sum_{(S,S'):S\cup S'=X} \mathcal{T}_{S,\varepsilon} \overset{+}{\wedge} \mathcal{T}_{S',\varepsilon} + \sum_{(S,S'):S\cup S'=X} \mathcal{T}_{S,\varepsilon} \overset{+}{\wedge} \mathcal{T}_{S',\overline{\varepsilon}} + \sum_{(S,S'):S\cup S'=X} \mathcal{T}_{S,\overline{\varepsilon}} \overset{+}{\wedge} \mathcal{T}_{S',\varepsilon}$$

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Example: combinatorial specification

Trees recognizing the letter a and no other letter, and not recognizing ε

$$\mathcal{T}_{\{a\},\overline{\varepsilon}} = a + \underset{\mathcal{T}_{\{a\},\overline{\varepsilon}}}{\bullet}\!\!\begin{array}{c}\mathcal{T}_{\emptyset,\varepsilon} + \mathcal{T}_{\{a\},\varepsilon} \\ +\mathcal{T}_{\{b\},\varepsilon} + \mathcal{T}_{\{a,b\},\varepsilon}\end{array} + \underset{\begin{array}{c}\mathcal{T}_{\emptyset,\varepsilon} + \mathcal{T}_{\{a\},\varepsilon} \\ +\mathcal{T}_{\{b\},\varepsilon} + \mathcal{T}_{\{a,b\},\varepsilon}\end{array}}{\bullet}\!\!\mathcal{T}_{\{a\},\overline{\varepsilon}}$$

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Fully reducible trees: a base case for output size

Definition (Fully reducible expressions)

A regular expression tree T is fully reducible when $\sigma(T) = \mathcal{U}$.

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- Size after reduction $p = |\mathcal{U}|$ for $T \in \mathcal{R}$.
- The class of fully reducible trees $\mathcal{R}$ satisfies the equation:

$$\mathcal{R} = \overset{\star}{\underset{\mathcal{T}_{\Sigma,\bar{\varepsilon}}}{|}} + \overset{\star}{\underset{\mathcal{T}_{\Sigma,\varepsilon}}{|}} + \overset{+}{\underset{\mathcal{R}\ \mathcal{L}}{\wedge}} + \overset{+}{\underset{\mathcal{L}\backslash\mathcal{R}\ \mathcal{R}}{\wedge}} + \overset{\bullet}{\underset{\mathcal{R}\ \mathcal{T}_{\varepsilon}}{\wedge}} + \overset{\bullet}{\underset{\mathcal{T}_{\varepsilon}\backslash\mathcal{R}\ \mathcal{R}}{\wedge}}.$$

$\Longrightarrow$ completes the combinatorial specification of $L(z, u)$.

Solving efficiently: auxiliary classes

- every tree : $\mathcal{L} = \bigcup_X \mathcal{T}_{X,\varepsilon} \cup \mathcal{T}_{X,\overline{\varepsilon}}$

$$\mathcal{L} = a + b + \varepsilon + \overset{\star}{\underset{\mathcal{L}}{|}} + \overset{\bullet}{\underset{\mathcal{L}\ \mathcal{L}}{\wedge}} + \overset{+}{\underset{\mathcal{L}\ \mathcal{L}}{\wedge}}$$

$$L(z) = 3z + zL(z) + 2z(L(z))^2$$

- trees recognizing ε : $\mathcal{T}_\varepsilon = \bigcup_X \mathcal{T}_{X,\varepsilon}$

$$\mathcal{T}_\varepsilon = \varepsilon + \overset{\star}{\underset{\mathcal{L}}{|}} + \overset{\bullet}{\underset{\mathcal{T}_\varepsilon\ \mathcal{T}_\varepsilon}{\wedge}} + \overset{+}{\underset{\mathcal{T}_\varepsilon\ \mathcal{L}}{\wedge}} + \overset{+}{\underset{\mathcal{L}\setminus\mathcal{T}_\varepsilon\ \mathcal{T}_\varepsilon}{\wedge}}\ .$$

$$T_\varepsilon(z) = \frac{z + zL(z)}{1 - 2zL(z)}$$

- trees not recognizing ε : $T_{\overline{\varepsilon}}(z) = L(z) - T_\varepsilon(z)$

The system becomes triangular

$$
\begin{aligned}
T_{\emptyset,\overline{\varepsilon}}(z) &= function(T_{\emptyset,\overline{\varepsilon}}(z)) \\
T_{\{a\},\overline{\varepsilon}}(z) &= function(T_{\{a\},\overline{\varepsilon}}(z), T_{\emptyset,\overline{\varepsilon}}(z)) \\
T_{\{b\},\overline{\varepsilon}}(z) &= function(T_{\{b\},\overline{\varepsilon}}(z), T_{\emptyset,\overline{\varepsilon}}(z)) \\
T_{\{a,b\},\overline{\varepsilon}}(z) &= function(T_{\{a,b\},\overline{\varepsilon}}(z), T_{\{a\},\overline{\varepsilon}}(z), T_{\{b\},\overline{\varepsilon}}(z), T_{\emptyset,\overline{\varepsilon}}(z)) \\
T_{\emptyset,\varepsilon}(z) &= function(T_{\emptyset,\varepsilon}(z), T_{\emptyset,\overline{\varepsilon}}(z)) \\
&\vdots \\
T_{\{a,b\},\varepsilon}(z) &= function(T_{\{a,b\},\varepsilon}(z), \texttt{and everyone above})
\end{aligned}
$$

- Each equation is of degree $2 \Rightarrow$ exactly solvable

$$T_{\{a,b\},\overline{\varepsilon}}(z) = \frac{1}{4z}\left(-\sqrt{\Delta(z)} + 2\sqrt{(2z+2)\sqrt{\Delta(z)} - 6z^2 + 2} - \sqrt{(2z+2)\sqrt{\Delta(z)} + 10z^2 + 2} - z - 1\right),$$

where $\Delta(z)$ is the determinant of the equation for $L(z)$.

Analytic step: square-root singularity

The expression

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implies a *square-root behaviour*

$$T_{\{a,b\},\overline{\varepsilon}}(z) \sim A - B\sqrt{1 - z/\rho}$$

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  $\Longrightarrow$ we show how to compute these efficiently.

Conclusion and further work

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Thank you!