Change of basis towards sources of zero entropy

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Work in progress with
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Meeting STIC-AmSud, RAPA2,
Online, 11 December, 2020.

Motivation: random generation

- Random generation from coin-tosses

  - Random binary digits $X_1, X_2, \ldots$ give uniform

$$X = (0.X_1 X_2 \ldots)_2 \in [0, 1]\,.$$

  - Discrete random variable Y simulated by

$$Y = k \Longleftrightarrow X \in I_k(\mathbf{p}) := \Big[\sum_{j<k} p_j,\ p_k + \sum_{j<k} p_j\Big)\,.$$

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- Simulation of an stochastic process

  - Sequence (source) of random variables $\mathcal{Y} = (Y_1, Y_2, \ldots)$
  - With $X_1, \ldots, X_n$, longest $(Y_1, \ldots, Y_m)$ obtainable ?
  - Digits (Y_i) of expansions of X in other basis $\longleftarrow$

Motivation: numeration systems

Natural question:

- Given t binary digits $d_1, d_2, \ldots, d_t$ of

$$x = (0.d_1 d_2 \ldots)_2 \in [0,1]\,.$$

- Number n of CFE-digits (partial quotients) deduced ?

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}\,.$$

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- rate of CFE digits per binary digit,
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Spoiler: there is $C > 0$, s.t. $n_t(x)/t \to C$ for almost every x.

Motivation: source transformation

General problem

- Classical: From source S_1 to S_2, both of positive entropy:

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  - Almost surely $L/t \to \infty$. More precise results?
  - Can we compare the two sources?

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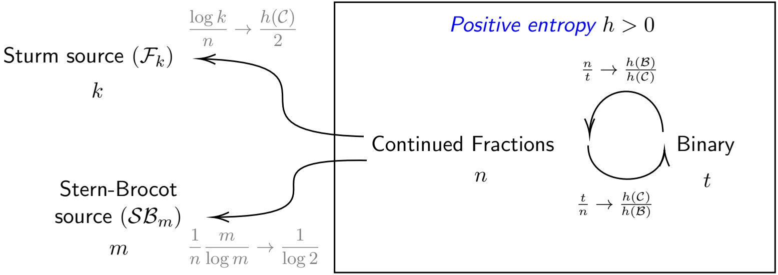
Theorem (BCRS'??)

If S_1 is the binary source:

- *For $S_2 = \texttt{Sturm}$, we have $\frac{1}{t}\log k \to \frac{\log 2}{2}$ for almost every x.*
- *For $S_2 = \texttt{Stern}-\texttt{Brocot}$, we have $\frac{1}{t}\frac{m}{\log m} \to \frac{6\log 2}{\pi^2}$ in prob.*

Our sources



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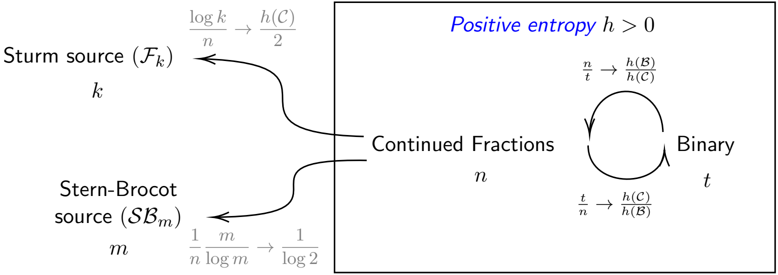

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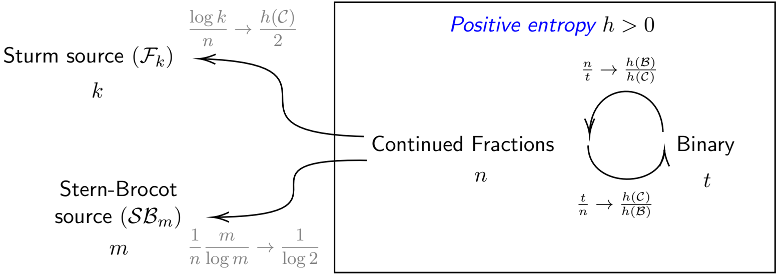

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- Continued fractions $x = [a_1, a_2, \ldots]$ as an intermediate source.
- For these cases, composing the arrows works $(*)$:
  - For Sturm $\frac{\log k(x,t)}{t} \to \frac{h(\mathcal{C})}{2}\frac{h(\mathcal{B})}{h(\mathcal{C})} = \frac{\log 2}{2}$ for almost every x.
  - For Stern-Brocot $\frac{1}{t}\frac{m(x,t)}{\log m(x,t)} \to \frac{1}{\log 2}\frac{h(\mathcal{B})}{h(\mathcal{C})} = \frac{6 \log 2}{\pi^2}$ in probability.

Plan of the talk

1. Sources of positive entropy
2. Important zero entropy sources
3. Conclusions

First historical results for $h > 0$

Theorem (Lochs '64)

The rate of CF-digits per decimal satisfies

$$\lim_{t\to\infty} \frac{n_{\mathtt{d}}(x)}{\mathtt{d}} = \frac{6\log 2 \log 10}{\pi^2} \doteq 0.9702701\ldots,$$

for almost every x.

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Theorem (Faivre '98)

$$\Pr\left\{x\in[0,1] : \frac{n_{\mathtt{d}}(x) - \mathtt{d}\times a}{\sigma\sqrt{\mathtt{d}}} \le \theta\right\} \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\theta} e^{-u^2/2}du\,,$$

where $a = \frac{6\log 2\log 10}{\pi^2}$ is the Lochs' constant, and $\sigma > 0$.

Expansions and intervals

- Decimal expansion $x = (0.d_1 d_2 \dots)_{10}$

$$I_{\mathtt{d}}^{\mathcal{D}}(x) = \left[(0.d_1 \dots d_{\mathtt{d}})_{10}, (0.d_1 \dots d_{\mathtt{d}})_{10} + 10^{-\mathtt{d}}\right].$$

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$$I_n^{\mathcal{C}}(x) = \Big[[a_1, \ldots, a_n, 1], [a_1, \ldots, a_n, 0]\Big] = \left[\frac{p_n + p_{n-1}}{q_n + q_{n-1}}, \frac{p_n}{q_n}\right],$$

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$$n_{\mathtt{d}}(x) = \max\{n \geq 0 : I_{\mathtt{d}}^{\mathcal{D}}(x) \subset I_n^{\mathcal{C}}(x)\}.$$

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- Take log of lengths : $-\log |I_{\mathtt{d}}^{\mathcal{D}}(x)| \approx -\log |I_n^{\mathcal{C}}(x)|$

Intervals: sources and partitions

Definition (System of partitions)

Sequence of topological partitions $\mathcal{P} = (\mathcal{P}_n)$ of $[0,1]$

- $\mathcal{P}_{n+1}$ refinement of $\mathcal{P}_n$ for every n.
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- *labeling functions* $\rho_n \colon \mathcal{P}_n \to \mathcal{A}$,
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unique coding $\varphi \colon x \mapsto (d_1 d_2 \ldots)_{\mathcal{P}}$ with $d_n = \rho_n(I_n^{\mathcal{P}}(x))$.

Entropy of a partition

Entropy dictates size of intervals

- *Shannon entropy*[1]:

$$H(\mathcal{P}) = -\lim_{k\to\infty} \frac{1}{k} \sum_{I\in\mathcal{P}_k} |I| \log |I| \ .$$

- *Point-wise*: for almost every x

$$h(\mathcal{P}) = -\lim_{k\to\infty} \frac{1}{k} \log \left| I_k^{\mathcal{P}}(x) \right| \ .$$

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Notation. For associated source: $I_n^{\mathcal{S}}(x)$, $H(\mathcal{S})$ and $h(\mathcal{S})$.

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Generalization Lochs': positive entropy

Lochs' index for sources $\mathcal{S}^1, \mathcal{S}^2$

$$L_n(x; \mathcal{S}^1, \mathcal{S}^2) := \max\{m \geq 0 : I_n^{\mathcal{S}^1}(x) \subset I_m^{\mathcal{S}^2}(x)\},$$

number of digits of $\mathcal{S}^2$ deduced from n digits from $\mathcal{S}^1$.

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Theorem (Dajani, Fieldsteel, 2001)

Consider sources $\mathcal{S}^1$ and $\mathcal{S}^2$, with defined and positive point-wise entropies $h(\mathcal{S}^1)$ and $h(\mathcal{S}^2)$. Then

$$\lim_{n\to\infty} \frac{1}{n} L_n(x; \mathcal{S}^1, \mathcal{S}^2) = \frac{h(\mathcal{S}^1)}{h(\mathcal{S}^2)}$$

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- **Binary.** Since $|I_t^{\mathcal{B}}(x)| = 2^{-t}$, $h(\mathcal{B}) = \log 2$.
- **Continued fractions.** Intervals satisfy $|I_k^{\mathcal{C}}(x)| = \Theta((q_k(x))^{-2})$

$$h(\mathcal{C}) = 2 \lim_{k\to\infty} \frac{1}{k} \log q_k(x) = \frac{\pi^2}{6 \log 2}.$$

Existence of point-wise entropy

Theorem (Shannon,McMillan,Breiman)

Let T be an ergodic measure preserving transformation on a probability space $(\Omega, \mathcal{B}, \mu)$ and let P be a finite or countable generating partition for T for which $H_\mu(P) < \infty$. Then for μ-a.e. x,

$$\lim_{n\to\infty} -\frac{\log \mu\left(P_n(x)\right)}{n} = h_\mu(T)\,.$$

Here $H_\mu(P)$ denotes the entropy of the partition P, $h_\mu(T)$ the entropy of T and $P_n(x)$ denotes the element of the partition $\bigvee_{i=0}^{n-1} T^{-i}P$ containing x.

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We recall that

$$h_\mu(T) = \sup\{h_\mu(T, \mathcal{A}) : \mathcal{A} \text{ countable partition of } X\}\,,$$

and

$$h_\mu(T, \mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} H\left(\mathcal{A}(U), \mathcal{A}(TU), \ldots, \mathcal{A}\left(T^{n-1}U\right)\right)\,,$$

where U is distributed according to μ.

Our zero entropy sources

Farey partition (Sturm source) and Stern-Brocot partition built by splitting intervals at *mediant*

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Farey partition $\mathcal{F}_k$:

- Base case: $\mathcal{F}_0 = \{[0,1]\}$.
- Building $\mathcal{F}_k$:
  split $\left[\frac{a}{b}, \frac{c}{d}\right] \in \mathcal{F}_{k-1}$,
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Stern-Brocot partition $\mathcal{SB}_m$:

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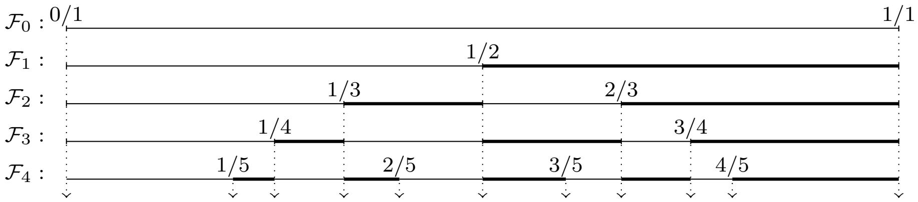

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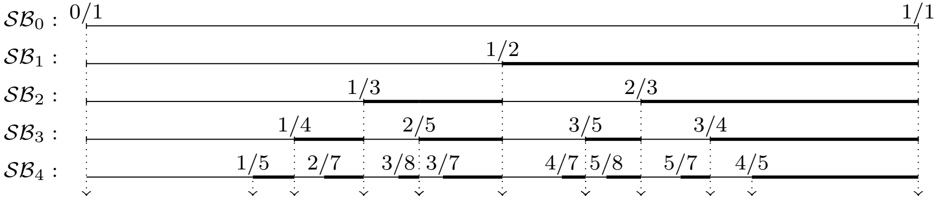

Farey,Stern-Brocot and Continued Fractions

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Lemma

Let $I_{r,n} = \left[\frac{p_{n-1}+rp_n}{q_{n-1}+rq_n}, \frac{p_n}{q_n}\right]$, $0 \leq r < a_{n+1}$ *:*

- *For Farey:* $q_{n-1}+rq_n \leq k+1 < q_{n-1}+(r+1)q_n$, $I_{r,n} = I_k^{\mathcal{F}}(x)$.
- *For Stern-Brocot:* $m = a_1+\ldots+a_n+r$, $I_{r+1,n} = I_m^{\mathcal{SB}}(x)$.

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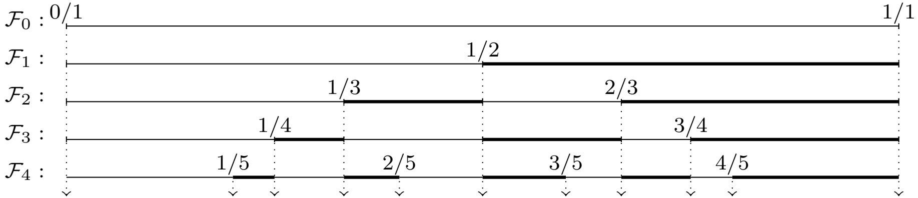

Properties:

- $\mathcal{F}_k$ determines char. Sturmian word up to $(k-1)$-th symbol.
- The end-points $\mathcal{F}_k$ are exactly $\{\frac{a}{b} \in \mathbb{Q} : 0 \leq a \leq b \leq k+1\}$.
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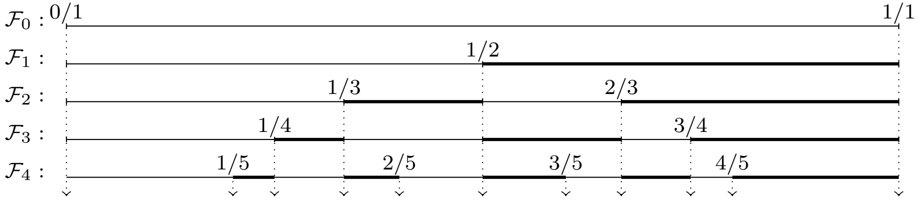

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Binary to Farey

Theorem (Lochs' index of Farey)

$$\lim_{n\to\infty} \frac{1}{n} \log L_n(x;\mathcal{B},\mathcal{F}) = \frac{\log 2}{2}\,.$$

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Lemma (Entropy of Farey)

For almost every x, for large $k \geq k_0(x)$

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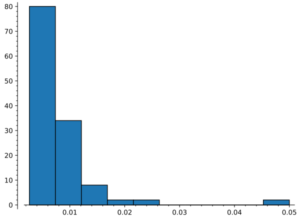

Figure. Histogram of interval sizes for $k = 20$. $\frac{1}{20^2} = 0.0025$, $\frac{1}{20} = 0.05$.

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Hand-waving argument: compare sizes

$$-\log \left| I_k^{\mathcal{F}}(x) \right| \sim 2 \log k \,, \qquad -\log \left| I_n^{\mathcal{B}} \right| = (\log 2) n \,.$$

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Two steps: lower limit and upper limit.

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- Intermediate source $\mathcal{B} \to \mathcal{C} \to \mathcal{F}$.

  "Triangle" inequality for transformation $\mathcal{S}^1 \to \mathcal{S}^2 \to \mathcal{S}^3$

  $$L_{j_n(x)}(x; \mathcal{S}^2, \mathcal{S}^3) \leq L_n(x; \mathcal{S}^1, \mathcal{S}^3)\,,$$

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- Upper limit follows from comparing sizes:

  $$I_n^{\mathcal{B}}(x) \text{ too big for } I_k^{\mathcal{F}}(x),$$

  when $k = 2^{n/2(1+\varepsilon)}$.

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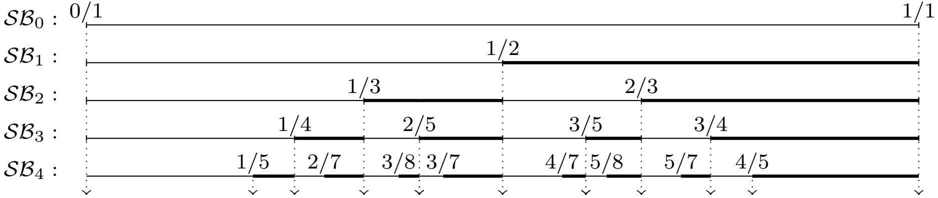

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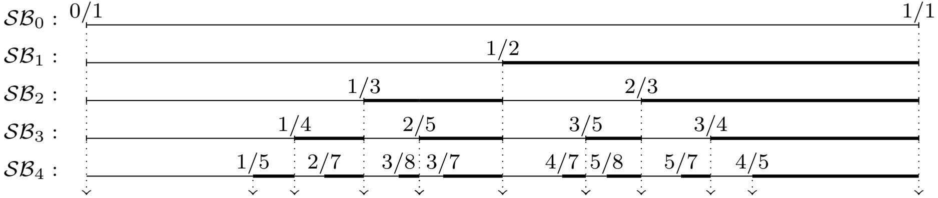

Associated to binary encoding of continued fractions:

$$[a_1, a_2, \ldots] \mapsto [0^{a_1-1}, 1, 0^{a_2-1}, 1, \ldots]$$

which follows construction of CFs by mediants.

Binary to Stern-Brocot

Theorem

$\lim_{t\to\infty} \frac{1}{t} \frac{m}{\log m} = \frac{6\log 2}{\pi^2}$ *in probability, where* $m = L_t(x; \mathcal{B}, \mathcal{SB})$.

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Proof.

Recall: depth is number of mediants taken $m = a_1 + \ldots + a_n + r$. $\square$

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Proof sketch for Stern-Brocot.

To use Khinchin, we use concentration [Faivre98] of $n = L_t(x; \mathcal{B}, \mathcal{C})$. □

Sums of partial quotients: why in probability?

Lemma

With probability 1, for $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n(\log n)^{1+\epsilon}} \sum_{i=1}^{n} a_i(x) = 0, \ \limsup_{n\to\infty} \frac{1}{n\log n} \sum_{i=1}^{n} a_i(x) = \infty\,.$$

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Partial sums $\sum_{i=1}^{n} a_i(x)$ are regular when we take out $\max a_i(x)$

Theorem (Diamond,Vaaler,98)

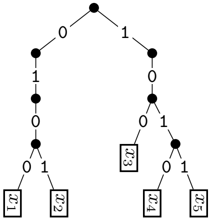
For large enough $n \geq N_0(x)$, *there is* $0 \leq \vartheta_+(n,x) \leq 1$ *such that*

$$\sum_{i=1}^{n} a_i(x) = \frac{1+o(1)}{\log 2} n\log n + \vartheta_+(n,x) \max_{1\leq i\leq n} a_i(x)\,.$$

Change of basis and tries: intuitions

Set of n words $x_1, \ldots, x_n$ emitted by $\mathcal{B} \Rightarrow$ *Trie* depth $t \sim \log_2 n$

$$\left|\begin{array}{c|c} x_1 & \mathbf{0100}101100\ldots \\ x_2 & \mathbf{0101}001101\ldots \\ x_3 & \mathbf{100}1101100\ldots \\ x_4 & \mathbf{1010}001001\ldots \\ x_5 & \mathbf{1011}111000\ldots \end{array}\right|$$



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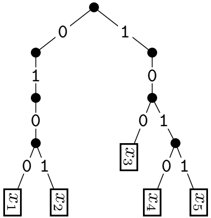

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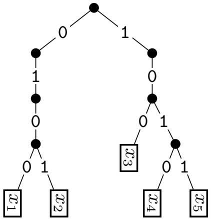

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$$m \approx \frac{6}{\pi^2}\log n \log\log n$$

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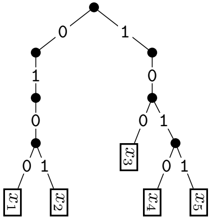

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Conclusions and further work

We have discussed two sources defined from the mediants

- The Farey partition $\mathcal{F}_k$
  - is seemingly irregular by construction, but
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3. Get a formal link with trie depth ?

Thank you!

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